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ABSTRACT

Investigations into children's number learning have been a feature of recent mathematics education research in Australasia. This book is a compilation of the major results of this research with the aim of making them more accessible to researchers, teachers, and others who may be able to use the findings to improve classroom practice. The introduction discusses worldwide trends in research on number learning and the Australasian contribution. The papers are divided into the following sections: The Development of Counting; Numeration and Place Value; Computational Processes and Strategies; and Computation and Number Sense. They vary in their mathematical focus, the age range investigated, the methodology employed, and their conclusions. Some common threads running through all their findings are discussed in the introduction. Contains 38 references. (JRH)

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Children's Number Learning

A Research Monograph of MERGA / AAMT

Edited by Joanne Mulligan
and Michael Mitchelmore

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The Mathematics Education Research Group of Australasia
The Australian Association of Mathematics Teachers Inc.

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Foreword

Gilah Leder

During the past decade, governments in many countries have spent considerable energy attempting to define, and restructure, school mathematics. Reflecting on conditions in England, Noss (1990) wrote:

> Incredible as it would have seemed only a decade ago, the mathematics curriculum of state schools, enshrined in 296 statements of attainment, is now part of the law of the land. (p.13)

While less prescriptive, governmental influence has also been felt in the United States of America:

> In 1990 the President of the United States and the National Governors Association announced their unprecedented agreement on national educational goals. For the nation to achieve those goals, it has become apparent that the American education system must be restructured. (Romberg & Wilson, p.1).

These writers pointed to the pioneering role played by mathematics. Indeed, work done by the National Council of Teachers of Mathematics (NCTM) in the curriculum and assessment areas has been widely regarded as an exemplary model for the other four disciplines (English, science, history and geography) singled out in the formal reform debate in the USA.

In Australia, governments have also turned to curriculum reform, with work in mathematics again leading the way.

> In April 1989 the State, Territory and Commonwealth Ministers of Education endorsed ten common and agreed national goals for schooling in Australia. As a result, the States and Territories have entered into the collaborative development of statements about the curriculum of Australian schools. This is

> the first such statement. Statements for other areas of the curriculum will follow. (Australian Education Council, 1991, p.i)

The mixed reaction, and ultimate rejection, of the national curriculum framework is now history. While reasons for opposing the national curriculum project varied, there is no question that research-based debates were influential. The tradition of documenting well-conceived and executed research within an Australian setting is continued in this book. Its publication is particularly timely, given the importance attached by the influential report *Educational Research in Australia* (McGaw et al., 1992) to further research in the areas of early childhood and mathematics education. Each contribution in this volume focuses on the intersection between these two areas.

The publication of this book also marks an important milestone for the Mathematics Education Research Group of Australasia Incorporated (MERGA). The editors, as well as the majority of the book's contributors, have strong links with MERGA, a critical sponsor and supporter of this venture. Selection of the Australian Association of Mathematics Teachers Inc. to be the book's publisher and distributor also cements a pleasing new development in the cooperation of the two organisations.

MERGA was formally established in May 1980, three years after holding its first conference (in Melbourne, in May 1977). From its inception, the Group has sought to

- encourage and promote quality research in all areas related to the teaching and learning of mathematics, particularly in Australasia;
- provide permanent means for sharing of research results and concerns among all members through regular publications and conferences;
- seek means of implementing research findings at all decision levels to the teaching of mathematics and to the preparation of teachers of mathematics; and
- maintain liaison with other organisations with similar interests in mathematics education or educational research. [MERGA Constitution, 1995, p.1]

There is no question that the contents of this volume are fully consistent with MERGA's aims of promoting and disseminating high quality Australian research. There is much of interest in the chapters that follow. May the contributions inspire healthy debate and further work! I commend the book with confidence and pleasure.

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April 1996

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Introduction

Michael Mitchelmore & Joanne Mulligan

Investigations into children's number learning have been a feature of recent mathematics education research in Australasia. Studies undertaken from a variety of theoretical perspectives, using different methodologies, and focusing on diverse topics across a wide age range have added considerably to our knowledge of how children's numerical cognition develops before and during schooling. Most of these investigations have been grounded in educational practice, in the sense that the studies have arisen from researchers' classroom experience, have been carried out in schools, and have been intended to help children learn. The editors of the present volume judged that the time was ripe to put the major results of much of this work together and to make them more accessible to researchers, teachers, and others who may be able to use the findings to improve classroom practice.

Our focus on a single area of children's mathematical development is in keeping with international trends. Number learning is particularly significant because number forms such a large part of every school mathematics curriculum. We do not, of course, wish to claim that numerical understanding develops independently of cultural or social influences, or without relation to other mathematical concepts. Hints of these influences and relations will be found here and there in this volume, but there is simply not space to consider them in full. Thus, although Australia and New Zealand are both multicultural communities, we have not been able to include studies of individual cultural groups.

The contributors to this volume have been asked to illustrate clearly the tasks they used with children and to explore the implications of their results for teaching practice. In this way, we hope to provide the reader with a clear and detailed picture of children's number learning and to encourage teachers and other researchers to experiment for themselves.

In this introduction, we shall firstly attempt to summarise recent international trends in research on children's number learning and to situate Australasian research in that context. We shall then briefly outline the content of each contribution and draw out the common threads. Lastly, we shall identify what we believe are the directions for future developments in this area.

Worldwide trends in research on number learning

The past fifteen years or so have seen a considerable increase in research on number learning all over the world. The growth has been marked by the publication of a huge number of research papers and several significant summaries of the findings of various research groups (e.g. Behr, Harel, Post, & Lesh, 1992; Bideau, Meljac, & Fischer, 1992; Harel & Confrey, 1994; Hiebert & Behr, 1988; Hughes, 1986; Leinhardt, Putnam, & Hattrup, 1992; Lesh & Landau, 1983; Steffe & Cobb, 1988).

The main characteristic of the recent wave of research has been a change of emphasis from children's performance to their thinking and understanding. The aim – to describe and explain children's mathematical behaviour – has not changed, but the focus has moved from analysing achievement scores on written tests of large samples to fine-scale investigations of individual children's strategies. In short, researchers have been trying seriously to understand number learning from the child's perspective. As a result, many researchers have been able to set up cognitive frameworks to explain how children understand number which hold much promise for guiding instructional practice.

Although many studies have continued to investigate children's strategies for carrying out abstract calculations, there has also been an increasing interest in children's methods of solving so-called "everyday" problems. There has been a general finding that children can solve simple word problems much earlier than had previously been presumed–using methods which they have usually not learned through formal schooling. Much effort has been put into classifying problems according to the different ways in which children process them as well as classifying children's strategies for solving them.

The recent literature on number learning can be classified into four areas: counting and numeration; addition and subtraction; multiplication and division; and rational numbers. The development of each area can be traced back to the work of a small number of researchers or groups of researchers. Seminal in the study of *counting* were the works of Steffe, von Glasersfeld, Richards and Cobb (1983) and Fuson (1988). Progress in this area was recently summarised by Fuson (1992). Much research on *addition and subtraction* stems from the work of Carpenter, Moser and Romberg (1982), and has also been summarised by Fuson (1992). The early work on young children's *multiplication and division* by Steffe et al. (1983) and Vergnaud (1983) has since been extended to include the understanding of multiplicative functions among older students (Greer, 1992; Harel & Confrey, 1994). Finally, research on *rational numbers* owes much to the pioneering work of Behr, Lesh, Post and Silver (1984).

The groups investigating number learning have adopted a number of theoretical perspectives but most of them have been strongly influenced by a resurgence of constructivism (see, for example, von Glasersfeld, 1991; Cobb, 1994). Thus many studies have assumed that children construct their own number knowledge or have investigated how this occurs through interaction with their physical and social surroundings. Others have followed an information

processing paradigm (Halford, 1993). The role of representation and visualisation in number learning has been another area of investigation (Goldin, 1992; Janvier, 1987) and the notion of an intuitive model has also been invoked (Fischbein, 1987).

Recent research on early number learning has had a considerable influence on the school curriculum and methods of instruction. For example, many of those listed above played a part in the development of the NCTM *Standards* (National Council of Teachers of Mathematics, 1989). More directly, some have investigated the implications for classroom practice (Davis & Maher, 1993) or have carried out direct classroom trials of their theories (Cobb, Wood, & Yackel, 1995); others have investigated the effect of helping teachers understand more deeply the early numerical development of their students (Fennema, Carpenter, & Peterson, 1989). A further, indirect result of the emphasis on children's strategies has been an increased interest in number sense as opposed to computational ability (Sowder, 1992).

The Australasian contribution

Australians' interest in mathematics education led to the foundation of the Australian Association of Mathematics Teachers (AAMT) in 1966 and the Mathematics Education Research Group of Australasia (MERGA) in 1977. Both organisations hold regular conferences and publish their proceedings (see, for example, Richards, 1995, and Atweh & Flavel, 1995) and between them produce two influential journals: the *Australian Mathematics Teacher* and the *Mathematics Education Research Journal*. A third organisation, the Mathematics Education Lecturers Association (MELA), has been influential at state and national levels. Many mathematics educators in New Zealand also participate in MERGA or their own New Zealand Association of Mathematics Teachers (NZAMT), and a number of researchers from the wider Australasian region are also members of MERGA. Australasian researchers are highly visible at international conferences and publish widely in refereed research journals; for example, at the 1992 International Congress on Mathematical Education (ICME), Australasians featured as presenters at all four subgroups of the Working Group on the Development of Elementary Mathematical Concepts. The extent of Australasian research may also be judged from the summaries produced for the last two ICMEs (Atweh & Watson, 1992; Atweh, Owens, & Sullivan, 1996).

Australasian research into the development of children's number concepts was given a substantial impetus by the 1984 ICME held in Adelaide, where several of the pioneers referred to above spoke about their work. They spawned a nucleus of followers "down under" who later visited their mentors in the USA or Europe for further research and returned to set up their own research groups. Subsequent research has been extensive (Perry, Mulligan, & Wright, 1992; Wright, Mulligan, Stewart, & Bobis, 1996) and has not only contributed to the international literature on children's number learning but also had a considerable practical influence in the region–for example, on Australian state syllabuses and the *National Statement*

(Australian Education Council, 1990), on the New Zealand National Curriculum (Ministry of Education, 1992), and on the professional development of teachers. Many of the researchers who have been most influential over the past twelve years are represented in the present volume, including some who have already written their own books (Hunting & Davis, 1990; English & Halford, 1995).

The content of the present volume

The arrangement of the papers in the present volume is based on the four areas of early number research referred to above, but with counting and numeration separated, addition and subtraction combined with multiplication and division, and a section on number sense added. However, because of the close connection between various aspects of number knowledge, there is a considerable overlap between the five parts. We shall first summarise each part and then consider the common themes which emerge.

Part 1: The development of counting

Previous research on children's basic understanding of number and counting has described various levels of development and suggested ways of helping children progress more rapidly from one level to the next. The three studies in this part all assess the effectiveness of classroom instruction based on the research indications.

Bobis applies the concept of number sense to the Kindergarten year, coming to the conclusion that it would be helpful for later development if young children could learn to break up sets of objects into smaller parts instead of simply counting them one by one. On the one hand, the smaller parts might be easier to count; most Kindergarten children can instantly recognise groups of up to four objects, and might learn to recognise larger groupings provided the objects are arranged in a regular pattern. On the other hand, the experience of partitioning would lead naturally into the concepts of addition and subtraction; knowledge of partitions of 10 would be particularly helpful in developing calculation strategies. The instructional activities fell into two categories: recognising the numbers of dots in small regular patterns, and recognising the number of filled and empty cells in "tens frames". These activities were used repeatedly throughout the year in two Kindergarten classes, supplemented by activities involving auditory patterning, counting on and counting back. Children's responses showed that they developed valuable methods of visualising numbers and their partitions which are very rarely achieved under current teaching methods.

One of the most active Australian researchers into early number learning has been Bob Wright, who in a series of projects starting with small investigations and leading up to a major professional development effort, has attempted to put the insights of Steffe et al. (1983) and Steffe and Cobb (1988) into practice. In his first chapter, *Wright* first summarises the earlier

research leading to several ways of classifying children's counting procedures and then describes a year-long test of experimental instruction in one Kindergarten classroom. The teaching activities were in many ways similar to those used by Bobis, but Wright's report focuses on the importance of increasing children's facility with forward and backward counting sequences. He shows how this facility can provide children with effective strategies for solving additive and subtractive tasks. By comparing assessments on five scales made at the beginning, the middle and the end of the year with assessments previously made in two schools following a standard Kindergarten program, Wright demonstrates clearly that children can achieve far more than was anticipated by the writers of current syllabuses and textbooks. The essential element of the new approach, Wright concludes, is the replacement of written tasks carefully chosen to be within children's assumed level of understanding by challenging activities which take them beyond that level and with no predetermined upper limit.

One outcome of Wright's work has been the Mathematics Recovery program described by *Wright, Stanger, Cooper and Dyson.* The main aim is to help "prenumerical" first grade students catch up with their peers so that they can benefit from the regular instructional program. The focus is on educating teachers about the general pattern of children's numerical development, and then in a very practical way helping teachers to learn how to assess children's stage of development and then to develop and deliver appropriate learning activities. (In its crucial involvement of teachers, the program has many points of similarity to Cognitively Guided Instruction (Fennema et al., 1989)). Wright et al. describe a first test of the program, in which six teachers each tutored four children. The overall results and the four case studies reported show that 16 hours of expert, individual tuition can lead to significant advances in numerical understanding. It is of interest to note that, since 1992, the Mathematics Recovery program has been used increasingly and with continuing success (Wright, Cowper, Stafford, Stanger & Stewart, 1994).

Part 2: Numeration and place value

As Wright (this volume) points out, children can advance far in their ability to solve simple arithmetical problems without having an understanding of the base ten structure of our number system. However, the numbers eventually get so large that a knowledge of place value becomes essential. The two studies in this part examine children's understanding of place value and conjecture on how it is affected by teaching.

Boulton-Lewis, working from an information processing perspective, shows that primitive counting relies on different structures ("relational mappings") than place value knowledge ("system mappings"). Operations requiring use of place value knowledge therefore impose a greater cognitive load and hence are more failure-prone. Teachers are frequently urged to use provide concrete materials to help children carry out the desired calculation but, Boulton-Lewis argues, children then need to process the extra mapping between the numbers and the materials and this can impose an even greater cognitive load. To test this conjecture, she interviewed a sample of children in Grades 1, 2 and 3 at the beginning and end of a school year

in which they used a variety of concrete materials for learning arithmetic. At each grade level, children improved over the year in their ability to give the value of each digit in a multidigit numeral. However – and this is a most surprising and disturbing result – only one half of the children showed any improvement in the way they represented large numbers using concrete materials; one quarter did not change and one quarter actually regressed from using a base ten representation at the beginning of the year to using individual units or a mixed representation at the end of the year. Boulton-Lewis concludes that it would be better for teachers to use one concrete representation to the point of utter familiarity rather than employing several representations "so that the children would not become bored".

Thomas looks at the development of place value knowledge across the entire primary school age range. In a cross-sectional study, he found that by Grade 6 children reach a fair degree of competence in identifying the value of each place in a numeral up to 1000 and in using place value knowledge in additive calculations but that performance for numbers with four or more digits is decidedly weaker. Further analysis showed rather poor understanding of the recursive nature of the base ten system: Most students did not propose groupings of groupings when asked to suggest how to group objects "to make counting easier" and could not make effective use of given groupings of groupings (many apparently not even recognising that a group of 10 groups of 10 comprises 100 objects). Thomas suggests that the concrete materials commonly used in school to teach place value, such as Dienes blocks, emphasise additive properties of a narrow range of numbers; very little is done in current practice to help students grasp the multiplicative structure of our numeration system as a whole.

Part 3: Fraction knowledge

All previous chapters have dealt with children's basic understanding of whole number. This section takes us on to rational numbers, looking at their experiential origins and their early manipulation.

Hunting sees the origin of fraction knowledge in the sharing experiences of pre-school children. He shows convincingly that, although subdivision tasks play a major role in children's early social life, the idea of equal subdivision emerges only slowly. Few everyday events involve exact subdivision, so that equal sharing is an idealisation or abstraction from many situations where sharing is approximately equal. Before they enter the regular school, children have had much experience of situations basic to the concept of one half and have learnt the process of dealing. Hunting shows how this pre-fraction knowledge can be built on during Grades K-2 through activities which make the equal-division aspect explicit, extend the ideas to other simple fractions, and introduce fraction notation. Two such activities are described in some detail: one based on a collection of farm animals and one using specially written computer software.

Pearn analyses Grade 3 children's strategies in solving various rational number tasks (several fraction tasks and related partitioning and ratio problems), all set in supposedly familiar

contexts. On the fraction tasks, about half of the children showed an understanding of one half which was strong enough to solve inverse problems; most of the others could only calculate a half of a given number of objects directly, and some could not even do this. Less than half of the children could calculate other unit fractions such as one third and one tenth. The partitioning and ratio tasks revealed frequent application of whole number calculation strategies, except in the more difficult cases where students often fell back on concrete modelling. Pearn argues for an earlier introduction to the vocabulary of simple fractions, in a variety of contexts, than is currently the case in primary mathematics teaching.

Part 4: Computational processes and strategies

The strategies children use in solving additive word problems have been mentioned in some of the earlier papers (Bobis, Thomas) in this volume. The three papers in this section investigate these strategies more closely and extend the discussion to multiplicative situations. All three papers also give many fascinating examples of children's thinking at various stages of development.

Cooper, Heirdsfield and Irons have been investigating children's mental computational strategies, both in the abstract and in problem solving contexts, in a longitudinal study of Grades 1 to 6. For this volume, they report on children's responses to additive and subtractive word problems in Grades 2 and 3. Previous research suggests a classification of mental strategies into a small number of well-defined categories; analysis of the transcripts obtained from videotapes of their children confirmed these strategies and added two more. The strategies children used were found to change considerably over the two years, both in sophistication and efficiency. Particularly striking in the Cooper et al. report are the range of strategies which children invent to do mental addition and subtraction, the increasing use of mental strategies based on the algorithms taught for pen and paper computation, and the power of some non-traditional strategies.

Mulligan and Mitchelmore report on a study of children's multiplicative strategies in which the Grade 3 children solved a variety of word problems through discussion among themselves and with the researcher, without direct instruction. The focus is on the many ingenious ways in which children represented the problems on paper and on their explanations of their solution strategies. The representations appeared to show the same intuitive models (direct counting, repeated addition, repeated subtraction, and multiplicative operations) found in earlier studies, but there seemed to be no direct relation between the intuitive models inferred from children's drawings, from their calculation strategies, and from the symbols children used to express their calculations. Mulligan and Mitchelmore's results make it plain that children's understanding can be many-sided, and that an adult (a teacher or a researcher) can easily misunderstand what a child understands if only one aspect of the child's behaviour is taken into account.

Outhred also uses children's drawings to explore their understanding, investigating two specific multiplicative situations: cartesian product and area. She shows that drawings made by children in Grades 1-4 who successfully solved cartesian product problems developed from the relatively unsystematic to systematic drawings clearly incorporating a one- or two-dimensional structure. However, as found by Mulligan and Mitchelmore (this volume), there was only a weak tendency for the children who drew more sophisticated representations to use more sophisticated calculation strategies. Drawings made by children who correctly solve area problems necessarily show a two-dimensional structure; more interest lies in the variety of errors made by those who have not learned this structure. Again, there was only a weak relation between the apparent sophistication of the representation and the calculation strategy. Outhred also attempts to find a correspondence between children's drawings and the cognitive models for cartesian product and area presented by Behr, Harel, Post and Lesh (1994). The match for cartesian product is much better than that for area, but the general conclusion is that the drawings seem to be expressing something different.

Part 5: Computation and number sense

The papers in this part take a wider look at children's numerical competence. They all include several aspects of numerical understanding together, and all investigate children's performance from a broad general viewpoint best described as "number sense".

Stacey and Groves consider the effect of calculators on children's number sense in Grades K-2. They conducted a large 4-year study in which a calculator was made available to each child and the teachers and researchers worked together to design and evaluate suitable learning activities. Calculators were readily accepted, being mainly used as a computational tool when teaching or reinforcing concepts and when solving problems. The examples which Stacey and Groves present show how students were able to tackle problems and explore topics which would have otherwise not have been easily accessible to them–in particular, large numbers, negative numbers and decimals. Systematic interviews and teachers' anecdotal comments confirmed that students had exceeded common expectations for their age groups in many aspects of numerical competence. Stacey and Groves interpret the results as showing that children can understand much more than is commonly recognised, especially when placed in a rich learning environment which the calculator can provide.

McIntosh summarises the results of three large studies of number sense in general and mental computation in particular. In the first study, students in Grades 2-7 were interviewed individually and their mental computation strategies analysed and classified. Some clear differences were found between the strategies used by more and less competent students. The second study, covering Grades 3, 5, 7 and 9, used written survey instruments to investigate students' attitudes to mental computation; the types of calculation students preferred to do mentally; and their mental computation performance. McIntosh found that students do relatively little mental computation in school and that many prefer not to carry out even elementary calculations mentally. The third study, also directed at Grades 3, 5, 7 and 9, used a written test

of number sense. Examples of students' responses to number, operations and application items revealed some widespread misunderstandings. The three studies have many implications for the teaching of arithmetic in schools.

English approaches number sense from a different direction by asking what types of situation children relate to various number sentences. Children in Grade 5 who were judged to be strong in number sense were presented with four simple equations (one each involving addition, subtraction, multiplication and division) and asked to create a word problem leading to each one. They were also asked to say which of four work problems they would most like to solve. The students seemed to see each type of number sentence as related to situations with a very narrow range of semantic structures; students who were judged to be strong in problem solving were only slightly more creative than the others. Sadly, even those students tended to prefer problems that appeared easy to solve. English deduces that children are not exposed to a sufficient variety of problem situations during the early teaching of arithmetic. As a result, the aspect of number sense which would see the same operation in various, superficially different situations would seem to be undeveloped at this age level.

In a study of 11-13 year olds, Irwin shows that children certainly do try to make sense of numbers, even if the sense they make is strongly influenced by their previous knowledge and can therefore be mathematically inappropriate. In her study, eight pairs of students interacted with each other and the investigator as they attempted to solve various novel problems involving decimal fractions; four pairs worked on contextual problems and four on decontextualised problems involving similar numbers. Three sessions produced significant increases in test scores, the increase being significantly greater for those who worked on contextual problems. Examples of student dialogues show how difficult they found it to integrate an unfamiliar structure (decimals) with the familiar structure of whole numbers. Nevertheless, several cases are recorded where a return to the meaning of decimal notation helped students to resolve the conflict between the old and the new. Other examples make it clear that a familiar context can help students recognise that a conflict exists, if not to resolve it.

Conclusions and implications

The papers in this volume vary in their mathematical focus, the age range investigated, the methodology employed, and their conclusions. But it is possible to discern some common threads running through all their findings.

Children's number learning

The various papers allow us to discern the important concepts which underlie children's number learning, and the relation between them. Fundamental is an understanding of counting which includes effective use of multiples of 10 and powers of 10 as markers (Bobis, Wright),

from which powerful informal methods of addition and subtraction can develop (Cooper et al.). Two other fundamental concepts are equal partitioning – on which the ideas of fractions (Hunting, Pearn) and division (Mulligan & Mitchelmore, Pearn) may be based – and equal grouping, the basis for multiplication (Mulligan & Mitchelmore, Outhred) and recursive grouping (Thomas). Place value, which in any case places a high cognitive load on children's processing capacity (Boulton-Lewis), involves all of these concepts and it takes a long time for the average student to master the standard notation for large and small numbers outside our common experience (Thomas, Irwin).

Despite the complexity of the ideas, children learn a great deal about number without any planned teaching. Even before entering school they learn to deal out equal numbers and can cope with simple uses of the fraction one-half (Hunting). Before being taught addition and subtraction, they invent some very sophisticated mental methods and hold on to them despite competition from formal algorithms (Cooper et al.). Before being taught multiplication and division, they learn how to solve multiplicative word problems (Mulligan & Mitchelmore). Some children discover for themselves how to calculate the area of a rectangle before instruction and even to solve cartesian product problems, which are not taught at all in primary school (Outhred).

Other chapters show how much more children can learn with the aid of planned instruction. Kindergarten children can learn to decompose numbers up to 10 (Bobis), count forwards and backwards with numbers up to 100 (Wright), and work with simple fractions (Hunting). "Prenumerical" Grade 1 children can catch up with their age peers (Wright et al.), and children in Grades K-2 can excel when allowed to use calculators freely (Stacey & Groves). Simply by discussing various ways of solving problems, Grade 3 children can learn to solve multiplicative problems (Mulligan & Mitchelmore).

Teaching about number

Several authors ask what it is about current teaching of arithmetic that seems to produce such disappointing results. Poor performance is documented for mental computation (McIntosh), arithmetic with concrete materials (Thomas), number sense (McIntosh), algebraic modelling (English) and decimals (Irwin). One explanation is that an emphasis on learning prescribed algorithms leaves little time for mental arithmetic (McIntosh) and stifles the development of powerful alternative methods based on a structural understanding of the number system (Cooper et al.). Computational skill with relatively small numbers can hide lack of conceptual understanding, so that the development of skills in handling larger numbers may be severely restricted (Thomas). Also, perhaps because arithmetical competence is regarded as a prerequisite for problem solving, the application of number knowledge in realistic situations may be seen as too difficult for young children and so may be limited to a narrow range of contexts or omitted altogether (English). The inhibiting influence of a prescribed curriculum is also evident (Wright, Stacey & Groves).

More worrying, certain aspects of current practice that were intended to increase understanding may in fact be having the opposite effect. In particular, the use of a wide variety of concrete materials to model place value may be detrimental to skill development in young children (Boulton-Lewis); such materials cannot in any case represent the general principle of place value needed in later development (Thomas). What seems to be important is not the materials themselves, but the understanding of number which children derive from them.

By contrast, what are the features of the successful teaching reported in this volume (and of the out-of-school learning which also seems to be so effective)? There seem to be five major components, all intimately related to each other. Firstly, the instruction takes full account of children's current experience, understanding and competence. Children are never set tasks they can solve without thinking nor those which they have no hope of solving. Secondly, children are challenged to extend their current understanding and competence in various ways– for example, by practice on routine tasks at a high level of difficulty to encourage the development of more efficient calculation strategies or through problem solving. Thirdly, no content is prescribed or upper limit imposed. There is always the flexibility to explore a new and unfamiliar idea, if the occasion arises. Fourthly, children are encouraged to develop their own methods and to gradually refine these, but are allowed to retain less effective methods for as long as they are needed. In other words, children are regarded as creative but inexperienced. Fifthly, children are expected to learn from each other as much as from their teacher. Cooperation is as important as competition. Overarching all these components is the crucial role of the teacher in assessing children's understanding, planning the next activities, and guiding children's investigations and interpretations. The teacher controls what happens in the classroom, not the textbook.

The future

There is now a substantial body of knowledge of how children learn about number, and how teachers might assist this learning. Although there are still many questions to be further investigated, we believe it is time to start putting this knowledge into practice. Doing so will provide many challenges for curriculum designers, teachers and teacher educators alike.

A first challenge is to design a curriculum sequence for primary school which takes account of the order in which it has been found children learn various number concepts. Efficient pen and paper calculation should be seen as part of the development of understanding, and not the be all and end all of arithmetical education. Links need to be made with previous knowledge, and children's innate problem solving abilities tapped more effectively. Some difficult ideas (like fractions) should probably be introduced informally at an early stage, and some new ideas (like recursive grouping) treated explicitly at a later stage. There is at least enough known already to allow a start on designing a valid sequence.

Secondly, teachers and their advisers will need to consider how to implement such a sequence without imposing just the type of prescription which it clear should be avoided. It is

no easy task to keep a general pattern of development in mind and at the same time allow for individual variations. An important goal would be the design of group and individual activities which can challenge children at various stages of development and help them make steady progress. The third challenge is then how to convince practising and preservice teachers that such a model of teaching could, in the long run, be much more effective and rewarding than the traditional one–and then to help them learn the necessary skills.

The most urgent need is therefore for implementation studies in which sequences are planned, taught, evaluated and refined. Both small and large scale investigations could, given close cooperation between researchers and teachers, have substantial outcomes of both a practical and a theoretical nature. Some chapters in this volume (Bobis, Mulligan & Mitchelmore) exemplify approaches which could be adopted by single teachers; others (Wright et al., Stacey & Groves) show what can be done on a larger scale with greater financial and institutional support.

Just as several researchers in the present volume conclude that there is no sense in which children can be said to have fully understood a concept, so shall we never fully understand children's number learning. We trust that the present volume will, however, have at least raised some questions which the reader will wish to answer.

Acknowledgments

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Part 1

The Development of Counting

Chapter 1

Visualisation and the Development of Number Sense with Kindergarten Children

Janette Bobis

Understanding of number is one aspect of mathematics that we expect schooling to develop. However, mathematics education research evidence confirms that an alarming number of students who experience traditional mathematics instruction lack the rich conceptual knowledge characteristic of a good "sense of number" (Markovits & Sowder, 1994). It seems that many children in upper primary grades are still struggling to construct meaning for numbers (Ross, 1986) and that the great majority of students continue through their secondary education demonstrating little understanding of how to solve number problems (Carpenter, Corbitt, Kepner, Lindquist & Reys, 1982; Lindquist, 1989; Threadgill-Sowder, 1984). Intervention studies attempting to develop number sense in upper primary and secondary school children have met with considerable success when introducing new concepts (Bobis, 1991a; 1991b; Markovits & Sowder, 1994), but highlight the difficulties of overcoming inefficient procedural strategies learnt prior to instruction.

This chapter draws together mathematics education research investigating aspects of, and the relationship between, number sense and visualisation. Until recently, research in the two areas has been quite separate and investigations involving very young children have been sadly lacking. The research study which is the focus of this chapter capitalises on the intuitive capacity of young children to form mental images. Instruction employed visualisation strategies to develop the ability in young children to recognise part-whole relationships of numbers and to use the numbers 5 and 10 as benchmarks – skills which are likely to be beneficial for young children as they develop flexible and creative strategies for dealing with numbers greater than 10 in later years.

Background

What is number sense and why is it important?

While rarely discussed a decade ago, the term "number sense" has since gained prominence in mathematics education literature and is now considered an essential component of the mathematics curriculum in numerous curricula from around the world (Australian Education Council, 1990; National Council of Teachers of Mathematics, 1989, 1995; Willis, 1990).

The following is typical:

> All people need to develop a good sense of number, that is, ease and familiarity with and intuition about numbers. This requires a sound grasp of number concepts and notation, familiarity with number patterns and relationships, a working repertoire of number skills and, most importantly, confidence in one's capacity to deal with numerical situations.
>
> (Australian Education Council, 1990, p.107)

Number sense is recognisable in students who seem to exhibit an "intuitive feel" for numbers, but it is difficult to define precisely and is almost impossible to assess accurately using traditional paper and pencil tests. Broadly speaking, it refers to a well organised conceptual network of number information that enables a person to understand numbers and number relationships and to solve mathematical problems in ways that are not bound by traditional algorithms. The National Council of Teachers of Mathematics (1989) identifies five components that characterise number sense: number meaning, number relationships, number magnitude, operations involving numbers, and meaningful referents for numbers and quantities. Such skill components are recognised as important because they lay the foundation on which more advanced skills later build and because they convince students that mathematics makes sense. Rather than being isolated skills, these five components are interrelated and seem to contribute to a general intuition about numbers that "develops gradually as a result of exploring numbers, visualising them in a variety of contexts, and relating them" in flexible and creative ways (Howden, 1989, p. 11).

Since the term was first used almost a decade ago, research interest in identifying and promoting number sense has steadily grown. Researchers have linked good number sense with skills observed in students proficient at mental calculation (Hope & Sherrill, 1987; Trafton, 1992), computational estimation (Bobis, 1991a; Case & Sowder, 1990; Rubinstein, 1985; Sowder & Sowder, 1989; Sowder & Wheeler, 1989; Trafton, 1992), judging the relative magnitude of numbers (Sowder, 1988), recognising part-whole relationships and place value concepts (Fischer, 1990; Ross, 1989) and problem solving (Cobb, et al., 1991). While individual studies have highlighted a variety of skills that invoke number sense, characteristics

repeatedly identified across many investigations include the flexible and creative use of strategies to solve number problems and the ability to judge the reasonableness of results. Taken as a whole, the various studies have resulted in a comprehensive list of attributes associated with the development of a good sense of number; but disturbing conclusions from the same investigations reveal that relatively few students manifest these characteristics.

The identification of such number sense components has allowed researchers to design instructional programs with the intention of developing such components with the aim of improving students' overall sense of number. For example, a study by Markovits and Sowder (1994) gave instruction to seventh grade students focusing on mental computation, number magnitude and computational estimation. For several months the students were exposed to daily lessons of about 10 minutes duration for each of the components. Findings revealed that skills indicative of number sense can be taught to students. A second important finding of the study and one which confirms the findings of previous studies (Bobis, 1991a; Porter, 1989; Wearne & Hiebert, 1988) was that "where robust knowledge and procedures already exist, such as for adding and subtracting relatively small numbers, there is less likelihood of change because the prior knowledge is already readily accessible" (Markovits & Sowder, 1994). It seems that in using activities designed to enhance number sense, it is more difficult to reteach familiar content than it is to introduce new content. Ross (1989) argues that this point alone makes it imperative for instruction conducive to the development of number sense to begin as early as possible. Unfortunately, research into the development of number sense in early childhood years has been sadly lacking.

Part-whole relationships

While not specifically addressing the issue of number sense, an instructional intervention study by Fischer (1990) designed to broaden Kindergarten children's understanding of number reported improvements typifying good number sense in older children. Fischer hypothesised that children learning the concept of number through the use of instruction emphasising a part-whole approach would be better prepared to understand place value concepts and would more easily comprehend numbers greater than ten than children receiving instruction based on the traditional curriculum characterised by a reliance on counting. Fischer found that early instruction emphasising part-whole number relations not only aided the development of basic number concepts, but also facilitated the children's ability to solve addition and subtraction word problems and to deal with place value, despite these applications not being specifically part of the instruction.

A part-whole understanding of number is an important development in children's mathematical achievement. Traditionally, K-1 classrooms focus on rote counting, reading and writing numerals and on activities involving the matching of number names to a set of discrete objects (one-to-one correspondence) (Department of Education, 1989). This emphasis on counting means that many students remain dependent on immature and often inefficient counting strategies to perform even the most basic computations. Although counting is

certainly an important skill (Wright, 1991), it is not conducive to developing good number sense. Counting a set of say five objects by ones does not help children recognise that the same set of objects can be decomposed into three and two or four and one. Interpreting number in terms of part-whole relationships makes it possible for children to think about number as compositions of other numbers.

Despite researchers' recommendations that instruction in number sense should begin early, research in this area with specifically designed curricula has been lacking (Van de Walle & Bowman, 1993). Recent interest in the capacity of children to use visual imagery to help them make sense of mathematics from an early age has provided mathematics educators with a basis from which they could design instruction specifically intended to develop attributes associated with good number sense. Before discussing visualisation within the narrower context of number sense for the early childhood years, it is necessary to first outline research pertaining to visualisation within the broader context of mathematics.

Visualisation and number sense

Visualisation, or visual imagery, has been of historical interest for more than a century and mathematics educators have been aware of its benefits to mathematical thinking for quite some time. Einstein remarked that his elements of thought were not words but "certain signs and more or less clear images which can be voluntarily reproduced or combined" (Gardner, 1983, p. 190). Given the increasing significance of visual media in modern times, it has been anticipated by some researchers that "to understand, analyse and predict, we will have to engage in some form of visual thinking" (Hershkowitz & Markovits, 1992, p. 38). Bishop (1989), in his review of research on visualisation, demonstrates that few students have difficulty in generating visual images and that this point alone makes it a very powerful tool in the hands of teachers who choose to use it.

At its simplest level, visualisation means "to form a mental picture of an object, a pattern, a formula or a shape" (Dawe, 1993, p. 61). When asked by a teacher to "imagine a star" or "think of a pattern you could make using these three shapes", each student will actively engage in some visualisation process, but individual processes may differ considerably, as may the final image–visualisation, it seems, is a very individual matter. Reynolds and Wheatley (1992), in an attempt to identify the ways in which Year 4 students relied on the use of visual imagery to make sense of mathematical tasks, found that each student utilised a rich but different use of imagery as they attempted to solve problems and to explain their mathematical thinking about problems. Similarly, Presmeg (1986a) concluded that imagery is useful in the construction of mathematical meaning and that it is still used widely by senior secondary students to solve mathematical problems. She identified five different kinds of visual imagery:

1. Concrete, pictorial imagery (pictures in the mind)
2. Pattern imagery (pure relationships depicted in a visual spatial scheme)

3. Memory images of formulae
4. Kinaesthetic imagery (involving muscular activity)
5. Dynamic imagery (moving)

Presmeg found that students with a greater relational understanding of mathematics tended to use more abstract forms of imagery such as dynamic and pattern imagery while students with less relational understanding tended to rely on concrete, kinaesthetic and memory images. Further evidence suggested that visual imagery was not utilised when students were required to implement prescribed, routine paper and pencil methods for solving computations. Such findings have been confirmed by other researchers (Brown & Presmeg, 1993; Brown & Wheatley, 1990; Reynolds & Wheatley, 1992, 1994; Reys, Reys, Nohda & Emori, 1995) and substantiate the premises on which classroom instruction utilising visual presentations such as concrete materials and diagrams are founded – that meaningful mathematics is image-based.

Until recently research pertaining to visualisation has remained quite separate from that of number sense, but researchers now recognise the importance of studying young children's visual representations of number in their attempts to explore their earliest understandings of number concepts (see Thomas, Mulligan & Goldin, 1994; Thomas & Mulligan, 1995; and the chapter by Thomas in this volume for further discussion of this topic). The possession of a good sense of number is undeniably beneficial to the understanding of more difficult concepts in later years, and the sooner it is learnt, the better. For classroom teachers, this raises the question of what specific activities can be structured for the K-1 level to promote the development of number sense. The use of concrete materials is important, but rather than moving directly from physical representations to the manipulation of abstract symbols to explain the traditional abstract procedures of algorithms, it is suggested that the emphasis be shifted to using visual imagery prior to the introduction of more formal procedures.

The following section discusses an instructional intervention research project specifically aimed at studying the effect on young children's developing number sense of visualisation strategies. Results and implications for teaching and learning are discussed and activities that can facilitate young children's understanding of number in the classroom are presented.

Visualisation and the development of number sense in early childhood

Background to the study

Research has shown that children as young as two years can learn to instantaneously recognise groups of two and three objects (Gelman & Gallistel, 1978) and that most children *by the age of four* are capable of instantaneously recognising groups of four objects. This process of instantaneous recognition of number patterns without counting is known as *subitising* (Folk, Egeth & Kwak, 1988). Having been surrounded by groups of objects from a young age, children develop stable mental images of these patterns and they are soon able to reconstruct them for themselves. Most people develop this visual skill and are hardly aware of its usefulness until they encounter another individual who has not developed it, such as a young child. For example, when a die is rolled during a game an older child or adult instantly recognises whether he or she has rolled a four, five or six. However, younger children and individuals inexperienced with the arrangement of dots need to count each dot before they can determine the number. Constant exposure to the number names used to describe the arrangements help children form pattern-name associations, although the very young may not fully understand the quantitative aspects of the patterns until later. To illustrate this skill, try the following exercise with a group of children of various ages.

Test one child at a time. Cover all the boxes in Figure 1.1. Start with the top row and reveal one box at a time for about one to two seconds each (short enough to prevent the child counting the dots). After each exposure, ask the child to tell you the number of dots in the box.

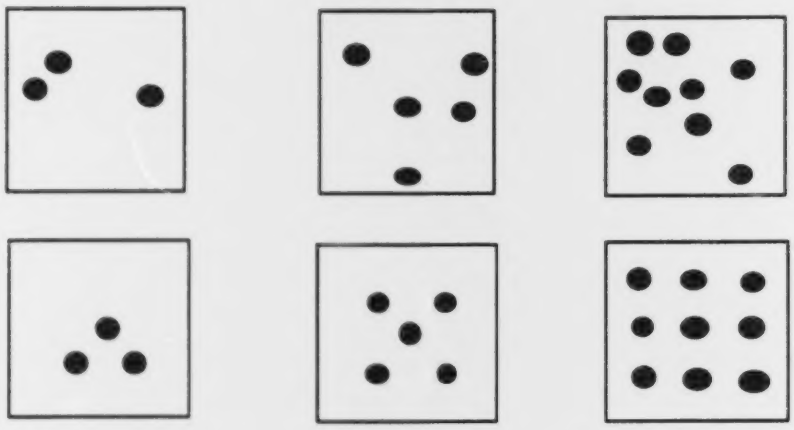


Figure 1.1

Which set of cards did children of various ages find easier to subitise? Why?

Organising the dots into familiar subgroups and patterns facilitates the ability to recognise and give number names to groupings, especially when the groupings are larger than five. Joining and separating visual patterns provides a basis for developing future number relations. For example, a child might recognise patterns for two and three in the five pattern. Given practice with dot patterns children can develop a considerable collection of set patterns which they recognise without counting. Recognising groups of two and three objects helps in seeing numbers four through ten as composed of smaller groups. This thinking involves a part-whole understanding of number.

Utilising children's intuitive visualising capacity can provide a basis for developing a knowledge of number relations which will provide a strong conceptual base for addition and subtraction strategies and enable children to be flexible and creative with mental computation, estimation and paper and pencil computation from a young age (Payne & Huinker, 1993; Reys et al., 1995).

The study

In the study reported here, Kindergarten children were encouraged to use visualisation strategies to mentally combine and separate numbers represented by arrangements of dots. It was predicted that by providing instructional activities and materials that were designed to utilise the intuitive skill of subitisation, children's understanding of part-whole number relations would be facilitated and hence their sense of number developed.

Method

Two mixed ability Kindergarten classes and their teachers from a suburban Sydney school participated in the year long study. The classes contained 32 and 35 children whose ages ranged from 4 years 5 months to 5 years 7 months at the start of the year. Boys and girls were equally represented in the former class but the girls slightly outnumbered the boys in the latter. Both teachers had more than five years' teaching experience at a variety of year levels.

Starting from week one of the school year, the classroom teachers focused on activities that utilised and developed visualisation strategies. These activities were integrated into various aspects of the ongoing mathematics program and were supplemented by auditory patterning, counting-back and counting-on activities. The teachers used sets of cards that represented the numbers from one to ten as an arrangement of dots. Except for the numbers one, two and three, each number was represented by at least three different dot arrangements. Figure 1.2 shows the cards that were presented for the number six.

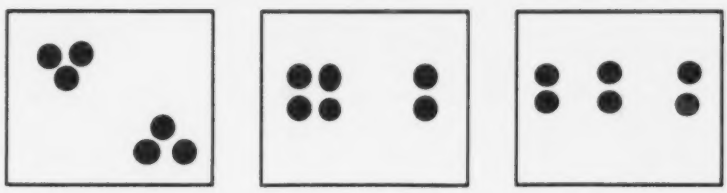


Figure 1.2

Dot cards utilise the ability of children to subitise and they encourage mental shifts from thinking about a number as one type of dot arrangement (or part-whole combination) to another. Activities with the cards were done with the whole class and with small groups of children working independently for approximately 10 to 15 minutes each day. Some of the activities used in the instructional sequence of the study are presented in Appendix 1.1. Children were encouraged to verbalise the visual patterns they saw for different numbers, and discussions concerning "nice" (that is, easy to visualise) patterns were encouraged regularly by the classroom teachers. The emphasis was initially on subitising skills using the dot cards for numbers less than five and was then extended to include larger numbers and activities with ten frames.

Ten frames are two-by-five rectangles into which counters or dots are placed to illustrate numbers less than or equal to ten (see Figure 1.3). This model for numbers was originally designed by Wirtz and is explained in greater detail by Labinowicz (1985) and Thompson and Van de Walle (1984). Like the dot patterns, the ten frame utilises the developing capacity of young children to subitise. Children learn to recognise patterns and their number names according to the number of dots and empty frames occurring on a flash card. For example, if the ten frame for eight was flashed, children were asked:

- How many dots are on the card?
- How many empty spaces?
- How many dots if we add one more (or less) to the number shown?
- How many more do we need to make ten?
- How many less to make five (or any other number)?

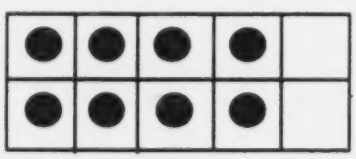


Figure 1.3

All the activities focused on identifying groups of numbers rather than counting one-by-one. They also required the children to focus on part-whole relationships of various number combinations – especially the decomposition of ten (see Appendix 1.2 for a summary of activities using the ten frame).

Other activities encouraged children to manipulate their mental images by adding or taking away dots to make a new number and then explain to a classmate how they got their new number using drawings, concrete materials and verbal descriptions. The overhead projector was used extensively by the classroom teachers to model these procedures to the class. After one arrangement for a number was given, children were immediately challenged to think of

another way to represent the same number. Probing questions such as, "What patterns or clues tell you that there are more than six dots?" and "Why did you move the dot over there?" were used by the teachers to elicit thinking strategies and visual cues used by the children to determine the number of dots.

The primary mode of obtaining information about pupils' mental images was through their verbal comments in class discussion and in task-based clinical interviews held in conjunction with their drawings and the instructional materials described above. Whole class sessions (lasting for about 10 minutes) and individual interviews (lasting for 3 to 5 minutes) were audio-taped regularly and transcribed throughout the year for each child so as to keep a developmental record of the children's responses. Samples of work were collected and often became the focal point for discussion during individual interviews.

Children's responses

In class discussions, children's descriptions of the visual images they developed for the specific groupings varied widely, being based on their own particular conceptualisation of part-whole relationships. Further, it was observed that as the pupils listened to each others' explanations, they began to think about the visually presented arrangements in more than one way and to independently elaborate on and extend their own ideas. For example, at first children recognised convenient arrangements of dots, usually in two groups. However, more able children could recognise multiple arrangements for the same number of dots, thus equating, say, six with three groups of two dots and with four and two dots. Another strategy that was used more frequently as the number of dots got closer to ten, involved the identification of the number of empty frames. Children were quick to recall that two empty spaces meant that there were eight dots in the ten frame. A more advanced strategy, noted in two children toward the end of the study, involved the use of mentally moving dots to make arrangements more convenient to subitise. While the use of strategies varied from child to child, the more able children used a wide range depending on the magnitude of the number and the arrangement of dots on the frame. The following excerpts illustrate some of the strategies children used to determine the number of dots on the ten frames as they were flashed before them.

Excerpt 1:

Teacher:	(Flashing a ten frame) How many dots did you see?.....Lara.
Lara:	Nine.
Teacher:	How do you know there are nine? You didn't have time to count all those dots.
Lara:	There is one empty space and there are ten spaces altogether.
Teacher:	How many more dots do we need to make ten dots?
Lara:	One more.
Scott:	How does she know that? How does she know that there are ten spaces altogether?

Teacher:	Lara?
Lara:	Because I counted them this morning when I was playing "Flash" with Tess.
Teacher:	Scott, you come and count them...

Excerpt 2:

John-Paul:	I know why that's a pattern for eight.
Teacher:	Come and show everyone then. (John-Paul moves to front and points to the ten frame.)
John-Paul:	See, there's four there and four there and that makes eight.
Teacher:	Very good John-Paul.
John-Paul:	I know another one.
Teacher:	We'd love to hear another.
John-Paul:	There's two spaces (pointing to the ten frame).
Teacher:	How does two spaces tell you that there are eight dots?
John-Paul:	(Pause)...Because...(Pause)..if there were two more there'd be ten.

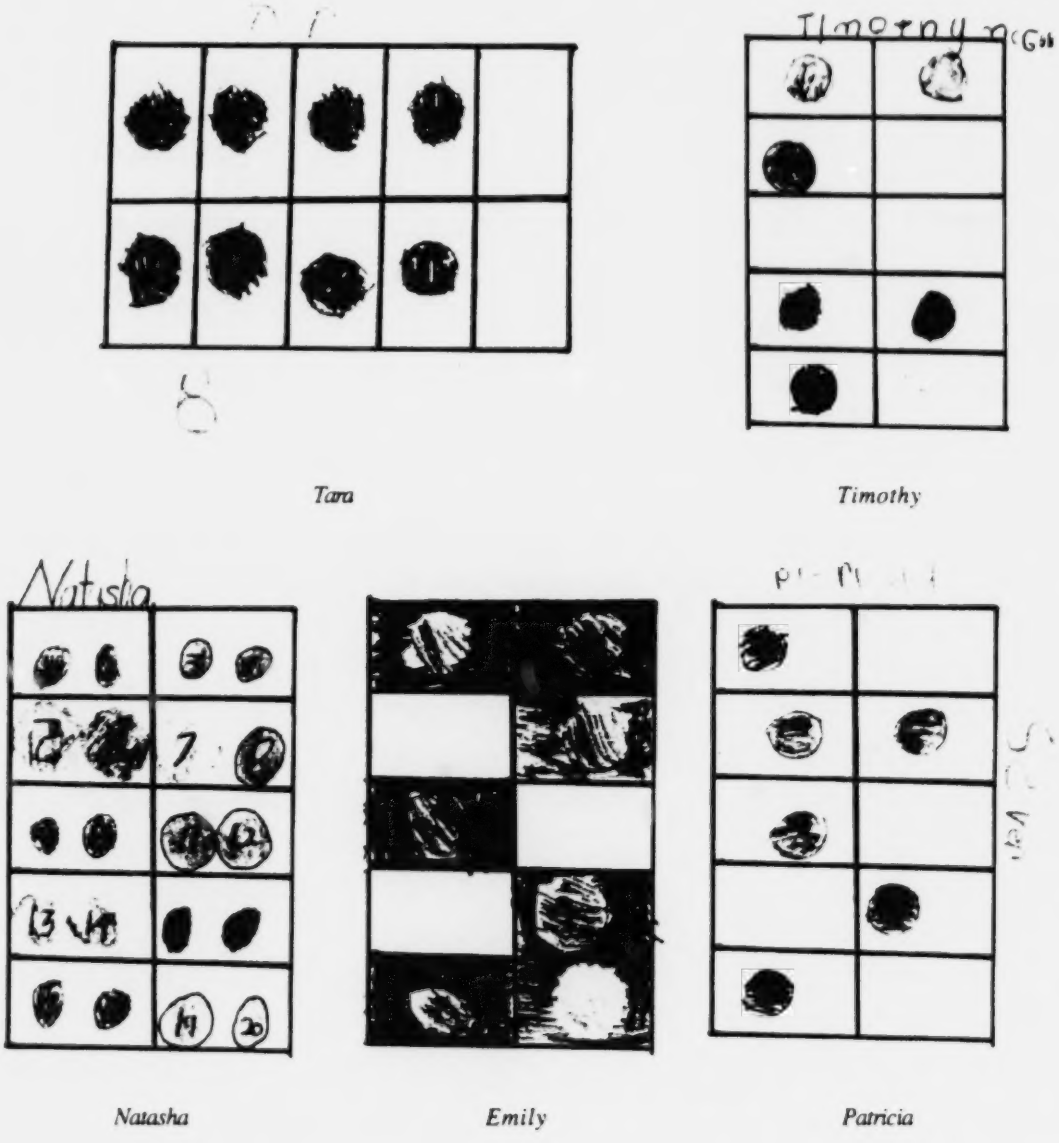
Although these children had not received any formal instruction in basic addition or subtraction facts, they were already seeing ten in terms of its parts.

"Seeing" patterns for numbers does not necessarily imply understanding, so it is imperative that mathematical activities require children to reflect upon their mental and physical representations (Presmeg, 1986a; Mason, 1992). In the present study, activities that encouraged the children to manipulate their mental images by adding or taking away dots to make a new number and then to explain how they got their new number to a classmate using drawings, concrete materials and verbal descriptions were considered a vital component of the instruction. For example, when asked to think of a special number and then to draw and discuss it, children were actively reflecting on their images and not just "looking" at them.

Although we were able to gain insights into students' mental activity through their verbal responses, their drawings also gave indications of how they thought of the visual patterns. Figure 1.4 contains examples of drawings made by children to record the "picture in their mind" of their special number. Comments accompanying the drawings were made after reading interview transcripts.

It can be seen how Tara and Timothy relied on the visual arrangement of dots to help them represent the mental image of their number. This type of imagery is akin to Presmeg's (1986a) first two types of visual imagery–pictorial and pattern imagery. Timothy decided to use a doubling pattern (3 and 3) while Tara used colour to distinguish her pattern of 4 and 4. She decided to draw this number on her ten frame because "I see two spaces and I know there must be 8 dots."

Natasha's representation for 20 is an example of how the brighter children may use what Presmeg (1986a) and others (Thomas, Mulligan & Goldin, 1994; Thomas & Mulligan, 1995) refer to as dynamic (moving) imagery to independently extend their own ideas to include number relationships beyond that traditionally taught at the Kindergarten level. She shut her eyes and "saw two frames but I only had one frame to colour-in so I put two counters in each space and made twenty."



Tara *Timothy*

Natasha *Emily* *Patricia*

Figure 4: Children's drawings of their mental images for numbers using the ten frame

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Emily's explanation of her ten frame revealed that she "saw" numerous part-whole relationships. She showed great flexibility in the way she visualised the various parts of her number : "...there are 3 spaces and 7 dots and there are 3 here, 3 here and 1 in the middle. And there are 2 here and 2 here and 1, 2, 3 on top".

Patricia's illustration did not immediately reveal any specific number relations. When questioned during an interview about her arrangement of dots it became evident that she did not understand the task and she simply placed counters randomly on the ten frame with no consideration to their arrangement. When asked to talk about any special patterns she saw, Patricia could only count the dots one-by-one. Although she still produced the correct number of dots, her unitary counting strategy was not conducive to developing part-whole relationships of numbers. This sample of work was collected during the third term in which the study was conducted and illustrates how some students require a great deal more exposure to these type of activities before they can comfortably form mental images that will aid their development of number. It was not until the beginning of the final term that Patricia could confidently arrange her dots to represent specific numbers that utilised part-whole relationships.

At the conclusion of the project, when the children had completed their first year at school, all the children participating in the study were able to represent numbers from 0 to 10 using the ten frame and to give explanations that revealed an understanding of part-whole relationships. Almost 80% of the children were able to visualise and construct with concrete materials numbers up to ten using three or more different part-whole arrangements, and 40% were able to extend this to numbers greater than ten (despite this not being explicitly part of the teacher instruction).

Implications for teaching and learning

Activities that focus on the visual identification of groups of numbers rather than counting one-by-one, such as those presented in this study, help develop understanding of part-whole relationships of various numbers–especially the decomposition of ten. The emphasis on encouraging children to be flexible and inventive with their thinking strategies is also important in promoting a good sense of number.

Given the mounting evidence that many students who have experienced traditional curricula lack the attributes associated with a good sense of number well into high school and even beyond, it is crucial that alternative instructional methods be explored. While counting is initially an important strategy used by children to deal with numbers, an emphasis only on counting does not allow children to develop a rich variety of number relationships. Unfortunately, the traditional primary school curriculum encourages instruction that moves directly from counting to the introduction of symbolic representation of addition and subtraction

procedures; activities conducive to the development of number sense are rarely provided during these crucial formative years.

Research highlighting the benefits of visualisation have provided classroom teachers with strategies useful for developing many skills and attributes associated with number sense from a very young age. Research reported here suggests that, with practice, even children as young as Kindergarten are capable of mentally combining and separating numbers when represented by a visual arrangement of dots, thus enabling them to recognise instantly the whole and its related parts – a very efficient strategy for aiding the recall of basic facts and for the development of mental computation skills. It has been shown that dot patterns and ten frame imagery are powerful representations that reinforce the development of many skills indicative of a student in possession of good number sense.

More time needs to be devoted in the Kindergarten and Year 1 mathematics programs to providing developmentally appropriate experiences that foster number sense. Intuitive skills like visualisation provide a powerful tool for teachers who choose to encourage it.

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Appendix 1.1: Activities for dot cards

1. Flash-and-tell
   Begin with dot patterns for numbers less than five. Add new ones to the familiar cards gradually. Flash a card and ask children to tell you what they saw. Encourage discussion that relates to the different combinations of parts. For instance, a pattern for six could be seen as two dots, two dots and two dots; or as three and three.

2. Flash-and-make
   Flash a card and ask the children to make the dot pattern with counters. Ask them to record how their pattern is arranged by drawing or tracing the counters. Different colours could be used to indicate different groups of dots.

3. Hide and seek
   Show part of a card with some dots hidden. Ask children to tell you how many dots are hidden if there are six dots altogether.

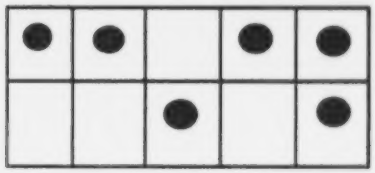
4. Pick a card
   Children pick a card from the pile and try to find a dot pattern that has the same number of dots, one more than or one less than the card picked.

5. Parts-wholes
   Provide children with parts of cards and ask them to find the rest of the card to make a particular number pattern.

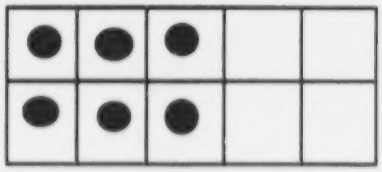
Appendix 1.2: Ten frame activities

1. Give children blank ten frames and ask them to make a pattern for any number they want using counters. Let them record all they know about their number in as many ways as possible e.g. drawing, writing, talking etc.

2. Ask the children to rearrange their number into another pattern. Try to encourage patterns that are easy to read without counting. For example:



a pattern for six

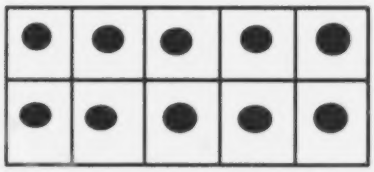


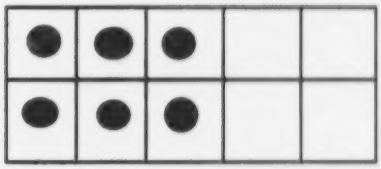
an easier to read pattern

3. Add one more counter (or remove one). How many spaces are filled now? How many are empty? What patterns do you see in your number? Do you see any doubles? What doubles do you see?

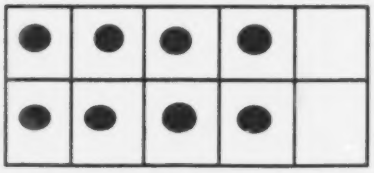
4. Close your eyes and picture your number pattern in your mind. What patterns do you see? Open your eyes and draw it from memory.

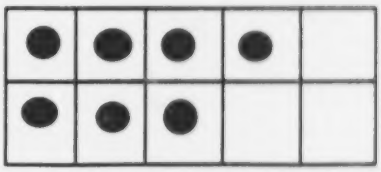
5. Develop the notion of a bunch of ten and some more (pre-place value activity) to enhance the visualisation of numbers between eleven and twenty e.g. ten and six make sixteen.





6. For number patterns involving 8 and 9 encourage the children to make ten and then add the remaining numbers. For example, for 8 + 7, make ten by adding 8 and 2, then add the remaining 5 dots to get 15. This should be done first with concrete materials (ten frames and counters) and then mentally. For example:





Take 2 dots from the seven dot pattern to make 10 with 5 remaining.

Chapter 2

Problem-Centred Mathematics in the First Year of School

Robert Wright

The notion that current curricula and classroom practices in mathematics in the first year of school seriously underestimate children's arithmetical knowledge is gaining credence. In recent years there have been in several countries major initiatives in redeveloping school mathematics curricula and, associated with these, new professional development courses to inform teachers about new curricula and teaching methods. In spite of these initiatives, it can be argued that no significant changes have occurred in teaching mathematics in the lower primary (i.e. infants) years and in particular, the first year of school. Curriculum emphases in early number and related topics have not undergone major changes in the last 30 years and have not changed significantly in the last 10 years (Wright, 1992, 1994b). Perhaps there is a general perception that schools have more or less "got it right" in mathematics at the lower primary level and this is why recent curricula revisions seem virtually to ignore the first two or three years of school. Related to this is the recognition that, beyond the lower primary years, problems with children's mathematical understanding require urgent attention and these should be the major focus of curricula revision and teacher development initiatives. On the other hand, one may argue that there are good reasons to assess much more closely what is taught and learned in lower primary mathematics.

In a recent article (Wright, 1994b) twelve issues in mathematics in the lower primary years were outlined and these need to be addressed. Of these, the following four are most relevant to this chapter.

- Current curricula and practice, by and large, have not taken account of the last 15 years of research into early number learning. See, for example, Fuson's (1992) recent review of research on learning addition and subtraction.
- There are serious doubts about the usefulness of prenumber and early number topics which, for a long time, have dominated mathematics in the first year of school – topics such as sorting, matching, classifying and cardinal numbers to 10 (Aubrey, 1993, p. 39; Wright, 1991c; 1992; 1994a; Young-Loveridge, 1989).
- Alternative topics which have been shown to be useful in advancing children's numerical thinking are available (Wright, 1991b).
- Teaching needs to be more informed by a knowledge of young children's arithmetical thinking and how it is likely to advance (Wright, 1994b).

Classroom-based research in lower primary mathematics has been undertaken since the mid-1980s and builds on programs of research into young children's mathematical learning – an area in which there have been major advances in the last 15 years (e.g. Carpenter, Moser & Romberg, 1982; Fuson, 1988; Steffe, von Glasersfeld, Richards & Cobb, 1983; Steffe & Cobb, 1988). Significant examples of classroom-based studies in lower primary mathematics are the work of Carpenter and Fennema (1992); Cobb and colleagues (e.g. Cobb & Bauersfeld, 1994; Cobb, Yackel & Wood, 1992; Yackel, Cobb & Wood, 1991); Fuson and colleagues (Fuson, Fraivillig & Burghardt, 1992); and Resnick (1992). In broad terms, each of these has generated useful models of classroom practice – models which address some of the current challenges in early mathematics instruction and provide alternatives or at least important refinements to current practice. Additionally, each of these may be viewed as a application of particular aspects of recent research into early arithmetical learning. Studies such as these address the important need for investigations which test in "real classrooms" the results of recent research into learning (Shulman, 1986). As well, these kinds of studies address the needs to integrate research on classrooms and research on students' ideas (Leder & Gunstone, 1990) and to integrate research on learning and research on teaching (Romberg & Carpenter, 1986).

The focus of this chapter is a year-long teaching experiment which involved a research team working with a Kindergarten class and its teacher. The study documents the progress of all 34 children in the class in terms of five models of particular aspects of early arithmetical knowledge.

Theoretical Background

The study reported in this chapter has its theoretical background in investigations by Steffe and colleagues of children's early arithmetical development (Steffe et al., 1983; Steffe & Cobb, 1988) and related work by the author (Wright, 1990; 1991a). Detailed overviews of this work are already available (Wright, 1991b; 1992; 1994a). In summary, young children's arithmetic knowledge is understood largely but not exclusively in terms of the following aspects:

(a) facility with number word sequences forward;

(b) facility with number word sequences backward;

(c) ability to identify numerals;

(d) subitising (von Glasersfeld, 1982), that is, the ability to ascribe a number instantaneously to a pattern of objects; and

(e) the relative sophistication of counting and other strategies used to solve what adults would regard as simple addition or subtraction problems;

(f) in the case of the more advanced students, their understanding of tens and ones.

Each of these aspects gives rise to a classification in terms of levels or stages. Five of these classifications, relating to (a) and (b) above, are used in this study. Two of these are associated with facility with number word sequences (relating to (a) and (b) above), one with ability to identify numerals (relating to (c) above) and one with subitising (relating to (d) above). These are identical to those used in an earlier study (Wright, 1994a), and are outlined in Tables 1 to 4. The fifth model is associated with counting and other strategies, is regarded as the primary or foundational model and relates to (e) above. This model is labelled the Stages of Early Arithmetical Learning model and is summarised in Table 5. More extensive explanations are available (Steffe, 1992; Steffe & Cobb, 1988; Steffe et al., 1983; Wright, 1991b, 1994a).

Arithmetical Stage 1 – Perceptual counting

Children at Stage 1 are labelled counters of perceptual unit items (Steffe et al., 1983, pp. 22-23). These children are limited to counting items they can perceive.

Arithmetical Stage 2 – Figurative counting

Children at Stage 2 are labelled counters of figural, motor or verbal unit items (Steffe et al., 1983, pp. 50-66). The Stage 2 child "typically counts from one in problem-oriented contexts in cases where, from an adult perspective, this is not necessary. The progressions from figural to motor, and motor to verbal represent progressively less dependence on sensory experience" (Wright, 1994a).

Arithmetical Stage 3 – Initial number sequence

The child at Stage 3 typically counts on, rather than counting from one when solving tasks involving hidden items. Children at Stage 3 are labelled counters of abstract unit items and are said to have constructed numerical concepts (Steffe et al., 1983, pp. 66-73). These children count on to solve additive and missing addend tasks and may use counting-down-from strategies (e.g. 17-3 as 16, 15, 14 - answer 14) but not counting-down-to strategies (e.g. 17-14 as 16, 15, 14 - answer 3).

Arithmetical Stage 4 – Advanced number sequence

Steffe posits two stages beyond the initial number sequence (Steffe 1992, pp. 95-6; see also Wright, 1994, pp. 27-8): the tacitly- and explicitly-nested number sequences. For the purposes of this study these two are combined. The terms tacitly- and explicitly-nested number sequences can be explained as follows. Consider the subtractive task 18-14 presented using a collection of 18 counters from which 14 are removed and both collections (i.e. 14 and 4) are then concealed. A child who has an awareness of 14 as a unit contained in 18 but does not at the same time have an awareness of the remaining numerosity (i.e. 4) as a unit is said to have a tacitly-nested number sequence. The child who can, conceptualise the whole (i.e. 18) and the two parts (i.e. 4 and 14) has an explicitly-nested number sequence. Characteristic of this stage are choosing the more efficient of counting-down-from and counting-down-to strategies, and using strategies that involve procedures other than counting by ones. Thus in additive and subtractive situations, the Stage 4 child uses strategies such as compensation, using a known result, adding to ten, commutativity, and subtraction as the inverse of addition, or has explicit awareness of the "ten" in a teen number.

Radical constructivism

The work of Steffe and colleagues in developing a theory of how young children learn beginning arithmetic, involved extensive collaboration with von Glasersfeld, the foremost proponent of the epistemology of radical constructivism (von Glasersfeld, 1984, 1991). In the last 15 years there has been an increasing focus on constructivist interpretations of learning and teaching in mathematics and science education. In the case of mathematics education at least, much of this focus has its impetus in the work of von Glasersfeld. Because this chapter focuses on applying theory and instructional techniques that resulted from constructivist investigations rather than being a constructivist investigation per se, this epistemology is not discussed further (but see, for example, Wright, 1991a).

Table 2.1	Construction of Forward Number Word Sequences (FNWSs)
Level 0:	**Absence of FNWSs.** Cannot produce the FNWS from 1 to 20, allowing errors such as omissions.
Level 1:	**FNWS from 1 as an unbreakable string.** If errors such as omissions are ignored, the child can produce a number word sequence from 1 to around 20. The child cannot produce the number word just after a given number. Dropping back to 1 does not appear at this level.
Level 2:	**FNWS from 1 to 20 as a breakable string.** The child can produce the number word just after a given number but drops back to 1 when doing so.
Level 3:	**FNWSs as chains in the range 1 to 10.** Produces the number word just after a given number in the range 1 to 10 without dropping back, but typically drops back for number after 10.
Level 4:	**FNWSs as chains in the range 1 to 30.** Produces the number word immediately following given numbers in the range 1 to 30 without dropping back.
Level 5:	**FNWSs as chains in the range 1 to 100.** Produces the number word immediately following given numbers in the range 1 to 100 without dropping back.

Table 2.2	Construction of Backward Number Ward Sequences (BNWSs)
Level 0:	**Absence of BNWSs.** Cannot produce the BNWS from 10 to 1.
Level 1:	**BNWS from 10 to 1 as an unbreakable string.** Can produce the BNWS from 10 to 1 but cannot produce BNWSs from number words less than 10. The child cannot produce the number word immediately before a given number, and the "dropping back to 1" strategy is not available to the child.
Level 2:	**BNWS from 10 to 1 as a breakable string.** The child can produce the number word immediately before a given number up to 10, but typically drops back to 1 when so doing.
Level 3:	**BNWSs as chains in the range 1 to 10.** Produces the number word immediately before a given number in the range 1 to 10, without dropping back, but typically drops back to 1 for numbers after 10.
Level 4:	**BNWSs as chains in the range 1 to 30.** Produces the number word immediately before given numbers in the range 1 to 30 without dropping back.
Level 5:	**BNWSs as chains in the range 1 to 100.** Produces the number word immediately before given numbers in the range 1 to 100 without dropping back.

Table 2.3	Numeral Identification
Level 0:	Cannot identify some or all of the numerals in the range 1-10.
Level 1:	Can identify numerals in the range 1-10.
Level 2:	Can identify numerals in the range 1-20.
Level 3:	Can identify one and two digit numerals.
Level 4:	Can identify one, two and three digit numerals.

Table 2.4	Identification of Spatial Patterns
Level 0:	Cannot identify regular spatial patterns (e.g. domino) with 3 or more elements.
Level 1:	Can identify regular but not irregular patterns with 3 to 6 elements.
Level 2:	an identify regular and irregular patterns with 3 to 6 elements.
Level 3:	Can identify regular and irregular patterns with 7 or 8 elements.

The classroom-based teaching experiment which is the focus of this chapter involves applying theoretical results from earlier teaching experiments which were not classroom-based. These earlier teaching experiments involved longitudinal investigations of the numerical development of children taught in individualised teaching sessions. Steffe and colleagues completed two such teaching experiments in the early 1980s (Steffe et al., 1983; Steffe & Cobb, 1988). These involved teaching beginning first-graders several times per week in teaching cycles of 18 to 20 weeks' duration, over a two year period. In the mid-1980s, the current author applied the results of these earlier teaching experiments in a year-long teaching experiment involving weekly individualised teaching sessions with four kindergarten children (Wright, 1990, 1991a, 1991b). Collectively, these teaching experiments led to the determination of new instructional approaches for lower primary mathematics. Wright (1991b) provides a detailed overview of these instructional approaches and their relationship to the theoretical model of the development of children's arithmetical strategies referred to above. Additionally, many of these instructional approaches are described below under the heading "Instruction". A major aim of the research reported in this chapter was to test the usefulness of these instructional approaches in whole class settings.

Purpose of the study

This study documents the progress in early arithmetical knowledge of children in a Kindergarten class where the mathematics program (i) involved instructional approaches regarded as appropriate for advancing children's early arithmetical knowledge, (ii) included a collaborative, problem-centred approach to mathematics (e.g. Cobb & Bauersfeld, 1994; Yackel, Cobb & Wood, 1991). It also investigates the effectiveness of the instructional approaches used in the study in terms of children's learning. The term "early arithmetical knowledge" includes, but is not limited to, the following: knowledge of and facility with number words and numerals for numbers up to 1000; early additive and subtractive strategies, that is, counting and other strategies; and emerging knowledge of the tens and ones structure of the numeration system.

Method

The current study applied at the Kindergarten level (i.e. the first year of school) a research methodology used by Cobb and colleagues at second and third grade levels (Cobb & Bauersfeld, 1994; Cobb, Yackel & Wood, 1992; Yackel, Cobb & Wood, 1991). A team consisting of a researcher, a research assistant and a class teacher collaborated to plan and evaluate three lessons per week for 17 weeks of the 1992 school year. Teaching was undertaken by the class teacher only. In the first nine weeks, the instructional approach was predominantly whole class teaching. The following lesson format was introduced in the tenth week and used throughout the following seven weeks: an initial period of whole class teaching; an intermediate period during which children worked in pairs to solve arithmetical problems; and a concluding period involving teacher-led discussion of solutions. It was considered appropriate to pair children who were reasonably similar in ability. All of the whole-class teaching and teacher-led discussions were videotaped, as were all of the problem solving sessions of two pairs of children. This chapter focuses on the children's advancements as documented in the beginning, middle and end of year assessments.

Instruction

The 17 weeks of teaching were spread through the school year as follows: a seven week period in each of the second and third terms, and a three week period in the fourth term. In the state of New South Wales, where the study was undertaken, the school year commences in beginning February and concludes around mid-December. The school year is divided into four terms typically of ten weeks' duration, with one or two weeks' break between terms.

First teaching period – Term 2

As stated earlier, the instructional approach in the first teaching period was predominantly whole class teaching with children responding verbally. Activities were progressively introduced in an order approximating that in which they are listed below. In the early weeks the activities involved numbers mainly in the range 1 to 20. As the weeks progressed and for some of the activities (e.g. number word sequence activities), the range was extended to 100 and beyond.

(a) Number word sequence activities. These include saying sequences forward and backward and saying one or two number words before or after a given number word (e.g. "What are the next two numbers after eleven"?).

(b) Subitising activities. Using an overhead projector, the teacher momentarily displays a regular or irregular spatial pattern, for example, the dots in a domino five pattern. The children's task is to find the number of dots in the pattern.

(c) Spacio-motor patterns. The teacher displays an outline of a geometric figure (e.g. a hexagon) or says a number (e.g. six). The children's task is to represent the pattern or number by tracing it with a hand in the air.

(d) Auditory patterns. With children looking elsewhere, the teacher claps an auditory pattern. The children's task is to copy or count the pattern.

(e) Counting movements. The teacher moves an item (e.g. a pencil) several times across an overhead projector. The children's task is to the number of movements.

(f) Fours-, sixes- and tens frame. Using a rectangular grid (2 x 2, 2 x 3, or 2 x 5) on an overhead projector, the teacher places counters on some of the squares and displays the frame for a short period. The children's task is to describe what they have seen (e.g. three red and three blue counters).

(g) Numeral track. Children are asked to name the numerals in turn. Numerals may be displayed or covered.

(h) Additive tasks. Two collections of counters, one or both of which may be concealed, are presented using an overhead projector and children are asked how many counters in all.

(i) Row tasks. Using a row of dots (e.g. 20 or 30) children are asked to state the number of a dot one or two before or after a dot whose number is specified.

(j) Subtractive tasks. Various tasks are presented using counters on an overhead projector (e.g. a missing addend task in which five counters are displayed and three are concealed. The children's task is to find how many more counters are needed to make eight).

Second and third teaching cycles – Term 3 and 4

The activities used in the first cycle were also used in the second and third cycles. In the third week of the second cycle a lesson format was introduced and used throughout the remainder of this cycle and the third cycle. This consisted of an initial period of whole class teaching, a middle period of problem solving in pairs, and a final period of teacher led discussion of solutions to problems. Addition tasks were used in the second cycle (five weeks) and subtraction tasks in the third cycle (three weeks). These problems were presented in three different ways: using pictures of objects, using a combination of pictures and numbers, and using numbers alone. For all three types a horizontal format was used. Materials such as counters were available for children to use when solving the problems.

Assessment of early arithmetical knowledge

A procedure developed and refined by the author (Wright, 1993; 1994a) was used to assess the arithmetical knowledge of each child in the class at the beginning, middle and end of

the study. This procedure is interview based and interviews are videotaped for subsequent analysis. Analysis of the assessment interview includes determination of a level or stage on each of the five aspects referred to earlier.

The tasks used to elicit arithmetical strategies included counting visible collections; counting collections separated into two parts, with one or both screened; missing addend tasks in which the interviewer screens one part of a collection and tells the child how many in all; missing subtrahend tasks in which the interviewer screens a previously displayed collection and, after removing and displaying part of the collection asks the child how many remains under the screen; and finally, comparison tasks involving two screened collections. The forward number word sequence (FNWS) tasks include saying the number word after a given number word, as well as saying FNWSs. The backward number word sequence (BNWS) tasks are analogous to the FNWS tasks. In the case of the numeral identification tasks the interviewer displays a card on which the numeral is written and asks the child to say the number. Finally, the tasks based on spatial patterns (Steffe et al., 1983, p. 30) involve briefly displaying a card on which a regular or irregular pattern of dots is drawn. Examples of the tasks appear in the Appendix.

Because of the ages of the children involved in this study, Arithmetical Stages 4 and 5 were combined and no FNWS and BNWS tasks at Level 5 (1 to 100) were administered.

Results and Discussion

Table 5 shows, for each of the 34 children in the class, the levels on each of the five models, for the initial, middle and final interviews held in April, August and October respectively. The rank order of the children has been determined by considering the data from the October interview (the value appearing on the right in each cell) as follows: the Arithmetical Stage is considered first; then the other levels are considered, in order from left to right.

Comparison with results from an earlier study

In 1990, the author undertook a study of numerical development in the Kindergarten year (Wright, 1994a). This included assessing at the beginning, middle and end of the school year the numerical knowledge of approximately half of the children in each of two Kindergarten classes following a regular instructional program. Tables 6 and 7 show, in a way similar to Table 5, the assessed levels on each of the five aspects. The schools in the 1990 study are government schools of average size, serving populations in one part of a city (pop.

approximately 40 000) and a nearby town. The school in the 1992 study is a non-government school drawing pupils from a wide range of socio-economic levels from throughout the city. Although precise comparisons between the 1992 school and the 1990 schools are difficult, the 1992 school is broadly similar to the 1990 school serving the area considered to be of higher socio-economic status. The results from the 1990 study therefore provide a basis for determining the relative progress of the children in the 1992 study.

Table 2.5	Arithmetical stages
Level 0:	**Preperceptual counting.** When attempting to count, the child is unable to coordinate number words with items.
Level 1:	**Perceptual counting.** Children are limited to counting perceptual unit items, that is, items they can perceive.
Level 2:	**Figurative counting.** Children count figural, motor or verbal unit items. The Stage 2 child typically counts from one in problem-oriented contexts. The progressions from figural to motor, and motor to verbal represent progressively less dependence on sensory experience.
Level 3:	**Initial number sequence.** Children at stage 3 are labelled counters of abstract unit items and are said to have constructed numerical concepts. They typically count on rather than counting from one when solving task involving hidden items. These children may also use counting-down-from strategies to solve subtractive tasks (e.g. 17 - 3 as 16, 15, 14 - answer 14) but not counting-down-to strategies (e.g. 17 - 14 as 16, 15, 14 - answer 3).
Level 4:	**Tacitly nested number sequence.** Stage 4 children are aware of the numerosity of part of a whole but not of the remaining numerosity. They can use counting-down-to to solve subtractive tasks, but typically choose the most appropriate of counting-down-to and counting-down-from.
Level 5:	**Explicitly nested number sequence.** Children can now conceptualise a whole (e.g. 17), a part (e.g. 14) and the remainder (e.g. 3) simultaneously. Stage 5 children typically use a variety of strategies other than counting in ones, such as compensation, using a known result, adding to ten, and commutativity.

Comparing progress on the Arithmetical Stages

In the 1990 study (Tables 6 and 7), of the 19 children who began at Stage 2 or lower, 11 (i.e. 58%) advanced one stage and 8 did not advance to the next stage. In the 1992 study (Table 5), of 30 children who began at Stage 2 or lower, 9 (i.e. 30%) advanced two stages, 13 (i.e. 43%) advanced one stage and eight did not advance one stage. Thus in overall terms, the progress of the 1992 children on the Arithmetical Stages notably exceeded that of the 1990 children.

Table 2.6		Assessments of children's numerical development during instructional period 1992 class				
Rank	Gender	Arithmetical Stage (Max. 4)	FNWS (Max. 4)	BNWS (Max. 4)	Numeral Ident. (Max. 4)	Spatial Patterns (Max. 3)
1	M	0 1 1	0 1 1	0 0 0	0 0 0	0 0 0
2	M	1 1 1	3 3 3	0 0 1	0 1 2	0 0 0
3	M	0 2 1	0 3 3	0 3 3	0 1 1	0 0 0
4	M	2 2 2	2 1 3	0 0 0	0 0 1	0 0 1
5	M	0-1 2 2	0 3 3	0 0 0-1	0 1 1	0 1 1
6	F	1 2 2	3 3 3	0 2 3	1 2 2	1 1 *
7	F	2 2 2	3 3 3	2 3 3	1 1 3	0 0 1
8	F	2 2 2	3 4 4	0 3 *	0 2 1	1 2 0
9	F	1 2 2	1 3 4	0 3 *	0 2 2	0 2 1
10	M	2 2 2	2 4 4	0 3 3	1 1 1	0 1 1
11	F	1 2 2	0 4 4	0 0 3	0 2 2	0 1 1
11	F	2 2 2	* 3 4	2 2 3	2 2 2	0 1 1
13	F	1-2 1 2	3 3 4	0 2 3	0 1 2	0 1 2
14	M	1 2 2	0 3 4	2 3 3	1 2 2	1 2 3
15	F	1 1 2	2 3 4	0 3 3	1 1 3	0 1 1
15	M	1 1-2 2	3 4 4	3 3 3	2 3 3	0 1 1
15	M	2 2 2	4 4 4	* 3 3	1 4 3	1 1 1
18	M	2 2 2	3 4 4	2-3 * 4	2 3 3	0 1 *
19	F	1 2 2	3-4 4 4	3 4 4	2 2 3	0 1 0
20	F	1 1 2	2 4 4	0 3 4	1 2 3	0 2 1
21	M	2 4 3	3-4 4 4	3 3 2	3 2 2	0 1 *
22	F	3 3 3	3 4 4	2 3 4	0 2 *	1 1 1
23	F	2 3 4	3 4 4	3 3 4	2 2 *	1 1 *
24	F	2 2 4	4 4 4	0 3 4	1 3 3	1 3 1
24	F	2 3 4	4 4 4	2 3 4	1 3 3	* 2 1
24	M	2 2 4	4 4 4	3 3 4	2 2 3	1 1 1
24	M	3-4 4 4	4 4 4	4 4 4	2 2 3	2 1 1
28	F	2 4 4	3 4 4	3 3 4	1 2 3	1 1 2
28	F	2 3-4 4	4 4 4	3 3 4	1 2 3	0 1 2
28	M	2 4 4	4 4 4	3 4 4	1 3 3	1 1 2
31	M	4 4 4	4 4 4	3 3 4	3 3 3-4	1 3 3
32	F	2 4 4	4 4 4	3 4 4	2 3 4	1 2 1
32	M	2 4 4	4 4 4	4 4 4	3 3 4	1 2 1
34	F	4 4 4	4 4 4	4 4 4	4 4 4	1 3 2

Note. Entries show the stages or levels on each aspect as assessed in April, August and October respectively. Entries in the form of a range (e.g. 2-3) indicate that the precise stage or level could not be determined. An asterisk (*) indicates that a datum is not available because of a technical failure.

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Table 2.7		Assessments of children's numerical development during instructional period (1990, lower SES class)				
Rank	**Gender**	**Arithmetical Stage (Max. 4)**	**FNWS (Max. 4)**	**BNWS (Max. 4)**	**Numeral Ident. (Max. 4)**	**Spatial Patterns (Max. 3)**
1	M	0 0 1	0 0 1	0 0 0	0 0 1	0 0 0
2	F	1 1 2	1 3 3	0 0 0	0 0 1	0 0 1
2	F	2 2 2	1-3 3 3	0 0 0	0 1 *	0 0 1
4	M	2 1-2 2	1 3 3	0 0 3	1 1 2	0 2 2
5	M	1 1 2	1 3 3-4	0 0 1	0 1 1	0 1 1
6	F	2 2 2	3 4 3-4	0 0 3	1 1 2	0 1 1
7	M	1 1-2 2	1 3 4	0 0 0	0 0 1	0 2 2
8	F	2 2 2	3 3 4	0 0 0	1 2 2	1 3 2
9	M	1 2 2	3 2 4	1 2 1-3	0 1 1	0 1 1
10	F	2 2 2	3 4 4	1 3 3	1 2 2	1 1 2
10	F	2 2 2	3 4 4	0 3 3	1 2 2	2 2 2
12	M	2 3 3	3 4 4	2 3 3	1 * 2	1 1 2
13	M	2-3 4 4	3 4 4	1-3 3 3	1 2 3	1 3 2
14	F	2-3 4 4	3-4 4 4	1 4 4	2 2 2	2 3 2
15	M	4 4 4	4 4 4	4 4 4	2 3 4	1 2 3

Note: Entries show the stages or level on each aspect as assessed in March, August and November respectively. An asterisk (*) indicates that a datum is not available because of a technical failure.

	Table 2.8		Assessments of children's numerical development during instructional period (1990, higher SES class)				
	Rank	Gender	Arithmetical Stage (Max. 4)	FNWS (Max. 4)	BNWS (Max. 4)	Numeral Ident. (Max. 4)	Spatial Patterns (Max. 3)
1		F	1 1 1	3 3 3	1 0 1-3	1 1 1	0 0 0
2		M	1 2 2	3 2 4	0 0 0	0 1 1	1 0 0
3		M	2 2 2	4 4 4	3 4 4	2 3 4	1 1 1
4		F	2 2-3 3	3 4 4	1-3 3 3	1 1 2	1 2 1
5		M	2 3 3	3 3 4	2 3 3	1 2 2	1 2 2
5		F	3 3 3	4 4 4	3 3 3	1 2 2	1 2 2
7		F	2-3 3 3	3-4 4 4	3 3 3	1 3 3	2 2 2
7		M	2 3 3	3 4 4	3 3 3	1 2 3	1 2 2
7		M	2 3 3	3 3-4 4	0 1-3 3	1 2 3	0 2 2
10		F	3 3 3	4 4 4	4 3 4	3 3 3	1 2 2
11		F	3 3 4	3 4 4	1-3 3 3	1 2 3	1 2 2
12		F	3 3 4	4 4 4	3-4 4 4	1 2 3	2 1 2
13		F	3 4 4	4 4 4	3 4 4	1 3 3	2 3 2

Note. Entries show the stages or levels on each aspect as assessed in April, August and October respectively. Entries in the form of a range (e.g. 2-3) indicate that the precise stage or level could not be determined. An asterisk (*) indicates that a datum is not available because of a technical failure.

Developing facility with Backward Number Word Sequences

One aim of the instruction in the 1992 study was to develop children's facility with BNWSs through instructional activities such as saying BNWSs and saying the number words before a given number. Comparisons across the 1990 and 1992 studies of children's initial and final BNWS levels indicate that this aim was clearly achieved. At the initial assessment, the distribution across the five levels in the 1992 study is similar to that of the 1990 study – approximately 40% were initially at Level 0 and less than 40% were initially at Level 3 or higher. In contrast, at the final assessment, the 1992 children are clearly more advanced than the 1990 children. In the 1990 study, of 28 children, 6 are still at Level 0 and 6 are beyond Level 3, while in the 1992 study, of 34 children, only 1 is at level 0 and 16 are beyond Level 3. These results show that instruction in BNWSs in the Kindergarten year resulted in almost all children being facile with BNWSs in the range 1 to 10 (i.e. Level 3), and around half of the children being facile in the range 1 to 30 (i.e. Level 4). The potential usefulness of developing BNWSs is discussed in the next paragraph.

BNWSs and early subtraction

In the 1992 study, 12 of 34 children were assessed at Arithmetical Stage 4 in their final assessment and all of these children had reached Level 4 on BNWSs. Consideration of the subtraction strategies used by these children suggests that their facility with BNWSs beyond the range from one to ten was important in their development of subtraction strategies. In their final assessments, these children solved written subtraction tasks with two-digit minuends and one-digit subtrahends (e.g. 36 – 3) as well as solving written subtraction tasks in the range 1 to 10. Typically solution strategies for the latter tasks involved sophisticated use of finger patterns. As an example, 9 – 3 was solved by simultaneously raising nine fingers, then simultaneously lowering three of the nine, and quickly saying six. In contrast, when solving tasks with two-digit minuends and one-digit subtrahends (e.g. 36 – 3), where a strategy using fingers would be inadequate, these children used a strategy involving counting back and keeping track of their counts. As an example, 36 – 3 was solved by saying, "thirty-five, thirty-four, thirty-three" and raising three fingers in turn to keep track. By and large, the strategies the children used to solve these tasks did not require an understanding of the tens and ones structure of the numeration system.

Progress in numeral identification

In the Kindergarten year, instruction on naming numerals typically focuses on numerals from 1 to 10 with extension to 20. In the 1992 study, naming numerals in the range 20 to 100 was a focus of instruction as well as numerals to 10 or 20. This instruction involved whole class teaching and displaying numerals for children to name. On some occasions this involved displaying a numeral sequence and on other occasions the numerals were displayed singly. A comparison of children's advancements in numeral identification in the 1990 study with that in the 1992 study reflects this additional emphasis. In the initial assessment, distribution across the Levels from 0 to 4 in the 1992 study is similar to that in the 1990 study. Around one quarter could not identify all numerals in the range 1 to 10 (i.e. they were Level 0), and less than 40% could identify all numerals in the range 1 to 20 (i.e. they were at least at Level 2). In the final assessment in the 1992 study, just over 50% could identify the numerals in the range 1 to 100 (i.e. they were at least at Level 3), and around 18% could not identify all numerals in the range 1 to 20 (i.e. they had not attained Level 2). In the final assessment in the 1990 study, around 32% could identify the numerals in the range 1 to 100 (i.e. they were at least at Level 3), and 25% could not identify all numerals in the range 1 to 20 (i.e. they had not attained Level 2).

The importance or usefulness of teaching numeral identification in the range 20 to 100, particularly to less advanced children (i.e. children who have not attained Arithmetical Stage 3),

remains an open question. As Steffe and Cobb argued (1988, p. 321), children's early arithmetic is, by and large, focused on the development of verbal number sequences. Thus it may be the case that knowing the names of two-digit numerals is not particularly significant or necessary for attaining Arithmetical Stage 4 if arithmetical problems are presented to children using number words rather than numerals.

Progress from Arithmetical Stage 2

A group of 12 children in the 1992 study can be characterised as follows in terms of their results in the final assessment: (a) they attained Arithmetical Stage 4; (b) they attained Level 4 on BNWS as well as FNWS; (c) they attained Level 3 or 4 on Numeral Identification (excepting one child for whom this result is unknown). In the initial assessment, only 3 of these 12 children attained Arithmetical Stage 3 while the other 9 attained Stage 2. Seven other children were at Stage 2 on the initial assessment, but none of these advanced beyond Stage 2 by the final assessment. There is a plausible explanation for the apparently surprising result that 16 of the 17 children initially at Arithmetical Stage 2 fall into two quite distinct groups. According to Steffe et al. (1993), children at Arithmetical Stage 2 are counters of figural, motor or verbal unit items. Further, figural, motor and verbal counting constitute a progression of distinct periods characterised by decreasing dependence on sensory material when counting to solve simple additive tasks. That children who are motor or verbal counters typically progress much more quickly to Arithmetical Stage 3 than do figural counters was established in teaching experiments conducted in the 1980s (Steffe et al., 1983; 1988; Wright, 1990; 1991a) and has been confirmed in results from the Mathematics Recovery project (Wright, Stanger, Cowper & Dyson, this volume). Thus a plausible explanation is that the 12 children who progressed to Arithmetical Stage 4 were initially motor or verbal counters and the seven who remained at Stage 2 were initially figural counters.

Children's overall progress

Collectively across the five models, 32 of the 34 children in the 1992 study made multiple advancements, that is, they advanced at least three levels or stages (the vast majority advanced significantly more that three). Of the remaining two children, one was initially at the lowest level on all five aspects and advanced two levels or stages only, and one was initially at the highest level on four of the five models and could not possibly have advanced more than two levels.

Conclusion

It is apparent from the results of the study that important gains were made by virtually all of the children over the course of the year. Particularly significant are the relatively large gains made by the children who were initially the most advanced. By comparison with children in the earlier study, it is reasonable to claim that these children made advancements significantly beyond those usually made by children in the Kindergarten year and that this can be attributed to the instructional approach. Significant features of this instructional approach are: (a) The incorporation of learning activities derived from the results of recent research into children's early arithmetical learning and not featured in current school programs and curricula; (b) a more challenging and extending approach to the teaching of number. (See Wright, 1994b for a review of curricula and teaching practice in early mathematics).

Wright (1992) discusses several points relevant to the need for a more challenging and extending approach to number. These include criticising "the dominance of prenumber and early number topics in the first year of school" (p. 134) and asking, "Has the justification for topics such as classification, ordering, one-to-one correspondence and cardinal and ordinal number been critically examined in recent years?" (p. 134):

> The primary role of prenumber and early number activities could result, at least in part, from a belief that children's early mathematics experiences in school should, initially at least, not involve problem solving or abstract mathematics. Alternatively it may be that, in many Kindergarten classrooms, mathematics lessons serve an important general purpose – that of initiating the children into processes of learning. ... The important goal for the teacher is for children to develop appropriate social behaviour rather than grapple with mathematical problems (p. 135).

In the same article, Wright criticises approaches which cultivate low-level strategies and inappropriate views of mathematical activity, namely overuse of activities involving structured materials and exercises based on pictorial representations.

The research reported in this chapter confirms that "learning and teaching lower primary mathematics continues to be an important area of focus and challenge for researchers and teachers" (Wright, 1994b, p. 33).

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Appendix 2.1

Examples of Tasks

Counting Tasks

Display five counters and ask the child to count them. Place out three counters screened from the child's view. Tell the child that there are three counters under the screen and ask how many counters in all.

Missing Addend Tasks

Display five green counters and screen three red counters. Say, "Here are five counters and there are some more under the screen to make eight altogether. How many are under the screen?"

Missing Subtrahend Tasks

Display a collection of ten counters and ask the child to count them. Ask the child to look away while you remove four counters and screen the remaining six. Remind the child that there were ten altogether and ask how many are under the screen.

Comparison Tasks

Display six red counters and ask the child to count them. Screen the counters. Similarly display and then screen four green counters. Use other counters to demonstrate the action of placing a green counter on each red. Say, "Remember there are six red counters under here and four green counters under here. If each green counter is placed on each red counter, how many red counters would not be covered with a green counter?"

Forward Number Word Sequence Tasks

Ask the child to start counting from one (stop after forty).

Ask the child to say the number word that comes straight after the number word said. Use number words in the ranges one to ten, eleven to thirty. Similarly, two or three number words after.

Backward Number Word Sequence Tasks

Ask the child to start counting from ten down to one.

Ask the child to start counting from twenty down to ten.

Ask the child to say the number word just before the number word said (Ranges 1-10, 11-30).

Numeral Recognition Tasks

Using a set of numeral cards, where each card containing one of the numerals below, ask the child to say the number displayed. The numerals and the order in which they are displayed are as follows: 4, 6, 7, 9, 12, 15, 10, 42, 57, 80, 66, 100, 341, 718, 206, 820.

Spatial Patterns

Using first the regular spatial patterns, and then the irregular spatial patterns, display each card briefly in turn and ask the child to say how many dots are on the card.

Chapter 3

First-graders' Progress in an Experimental Mathematics Recovery Program

Robert Wright, Garry Stanger, Margaret Cowper and Rosalie Dyson

Mathematics Recovery is a three-year collaborative research project jointly funded by the Australian Research Council and regional government and Catholic school systems in North-Eastern New South Wales. The project began in July 1992 in six schools and involved nine new schools in 1993, and an additional ten schools in 1994. In New South Wales, children's school commencement age typically ranges from 4.6 to 5.5 years and the school year runs from the beginning of February until mid-December. A detailed overview of the background and plan of the project has already been reported (Wright, 1992a). The major aim of the project is to establish an accredited, year-long, professional development course for teachers which has, as its focus, a program of early intervention in arithmetical learning. In this program, selected first-grade children (6- and 7-year-olds) are placed in a long-term, individualised teaching program which has the purpose of advancing their arithmetical knowledge to a level at which they are likely to learn successfully in a regular class. An additional aim is to establish an accredited course to prepare teacher-leaders to teach the professional development course. This chapter focuses on the first year of the study and outlines the instructional approach and activities, participants' progress and implications. The theoretical basis of this study has been outlined by Wright (this volume) and will not be repeated here.

Progress across the arithmetical stages in the first two years of school

Our work on current (e.g. Wright, 1992a; 1993) and earlier (e.g. Wright, 1992b; 1994) research projects in recent years has involved interviewing more than 200 children aged from three to eight years to assess their arithmetical knowledge. Several strong indications have emerged from these studies. These include the following.

(a) Reasonable educational goals for children are to reach Arithmetical Stage 2 or 3 at the end of the Kindergarten year and Stage 5 at the end of Grade One.

(b) Around 30% of beginning Kindergarten children are at or below Stage 1, and around 16% are at Stage 3 or beyond.

(c) The vast majority of children who begin Kindergarten at Stage 2 progress to at least Stage 3 by the end of the school year.

(d) The children who are at or below Stage 1 at the beginning of the year are unlikely to reach Stage 3 by the end of Kindergarten.

(e) Only a very small percentage (probably less than 5%) of beginning first grade children have not attained at least Stage 2.

(f) Also Grade One children who begin at lower than Stage 3 are less likely to advance to Stage 5 than are children who begin at Stage 3 or 4.

The children who begin the Kindergarten year at Stage 1 or below are by and large the children who begin first grade at Stage 2 or below. It is from this 30% of the age cohort that Mathematics Recovery candidates are drawn. As indicated by (e) above, only a small minority of these are likely to be at Stage 1. Nevertheless, as well as there being differences between the Stage 1 and Stage 2 children, there are notable differences amongst the children at Stage 2. Differences of the latter kind might relate to: (a) counting type – figural, motor or verbal (Steffe et al., 1983; 1988; see also Wright, 1989; 1991a); facility with forward or backward number word sequences (Wright, 1994); or ability to identify numerals (Wright, 1994).

Research questions

This chapter focuses on the progress made by the Mathematics Recovery participants in comparison with non-participating counterparts who were assessed as Stage 2 or below at the time of their initial interview. Specifically, this chapter addresses the following research questions:

(a) In terms of arithmetical stages, what progress was made by participating children over the eight-week teaching cycle?

(b) How does the progress of participating children compare with that of the non-participating counterparts who, at the beginning of the teaching cycle, were at Stage 2 or below?

(c) In terms of the four models of young children's numerical development, what progress can be made in a Mathematics Recovery program?

Additionally, this chapter includes a case study of each of four participants in the Mathematics Recovery program. Each case study documents a child's arithmetical knowledge prior to commencement of the teaching cycle and at its completion, and provides an overview of the child's Mathematics Recovery teaching sessions.

The study

During the period from July to December 1992, six teachers undertook a 20-week accredited professional development program, which, in organisational terms, resembled the Reading Recovery teacher development program (Clay, 1987; Pinnell, Fried, & Estice, 1990). This involved the teachers working on the Mathematics Recovery project during the morning half of each working day. The project focused on first grade children, that is, children in the second year of school. For each of the six teachers, the 20-week program included a six week orientation; four weeks which focused on assessing around 15 first grade children, selecting as project participants four children who were low attainers on the assessment, and planning teaching sessions for the participants; and an eight-week cycle during which each teacher individually taught four participants, four mornings per week, for 30 minutes per child each morning. On the fifth morning of each week the teachers attended an on-going professional development program. In the final two weeks all 15 children in each of the six schools were reassessed by their teacher permitting the comparison of the progress of the participating children with their non-participating counterparts.

Children's progress was assessed by individualised pre- and post-interviews conducted in each school by the project teacher. Aspects of young children's arithmetical knowledge assessed during each interview included learning stage in the construction of the number sequence; facility with forward and backward number word sequences; numeral identification; and knowledge of tens and ones. All interviews were videotaped for subsequent analysis. Detailed descriptions of this assessment procedure are available in Wright (1991c, 1993, 1994).

The teaching sessions

An important aspect of the project was the development of a bank of instructional activities for use in the Mathematics Recovery teaching sessions. Understanding the purposes and ways to use these activities was prominent in the teachers' professional development course. Each teacher used the bank of instructional activities to develop an individualised teaching framework for each of the four children she taught. The initial development of the teaching frameworks occurred under the guidance of the research team, as did weekly

redevelopments of the teaching frameworks. The latter was informed by the teachers' on-going reviews and evaluations of their teaching sessions. All of the teaching sessions were videotaped, and these videotaped records together with written notes made during and after teaching sessions were used by the teachers in their evaluations. Review and evaluation of teaching sessions were provided in the weekly professional development meetings during which teachers reported on their teaching using videotaped excerpts. Additional review and evaluation occurred during individual school visits by a member of the research team. Participating teachers maintained professional journals throughout the study.

The instructional approach

The instructional approaches of the Mathematics Recovery project were adapted from approaches used in earlier constructivist teaching experiments (Steffe et al., 1983; Steffe & Cobb, 1988; Wright, 1989, 1991a, 1991b) and include presenting children with arithmetical tasks which are problematic and challenging for them. These tasks are designed to be within or just beyond the range in which the child currently is able to solve tasks. Underlying this is a belief that these kinds of instructional settings are the ones most likely to lead to cognitive reorganisation, that is, modification of current schemes. One reason why the instructional approaches developed in the earlier studies were considered useful for Mathematics Recovery is that the participants in the Mathematics Recovery project are similar to the children who participated in the earlier studies – amongst the least advanced at their class level. Additionally, the Mathematics Recovery participants are similar in age and class levels to the children in the earlier studies.

Instructional activities

The instructional activities were adapted from those used in constructivist teaching experiments (e.g. Steffe et al., 1983; Steffe & Cobb 1988; Wright, 1989, 1991a, 1991b) and included: (a) additive and subtractive tasks involving collections or rows of counters some or all of which are hidden; (b) activities involving copying or counting sequences of sounds or movements; (c) activities using numeral cards arranged either singly or in numeral tracks or grids; (d) ascribing number to spatial patterns which are briefly displayed; (e) activities using collections of tens and ones; (f) number word sequence activities such as reciting sequences and stating number words before or after a given number word. The instructional activities described by Wright (this volume) are indicative of those used in this study.

Individualised instruction

As in the constructivist teaching experiments referred to earlier, and in Reading Recovery, individualised rather than group instruction was used throughout the Mathematics Recovery project. In the case of Reading Recovery, empirical evidence in support of individualised instruction is available (Pinnell, Lyons, DeFord, Bryk, & Seltzer, 1991). The question of the relative advantages of individualised versus group instruction is a complex one. On one hand it could be argued that withdrawal from the classroom for individualised instruction is at odds with current emphases on mainstreaming and collaborative learning. On the other hand, Mathematics Recovery participants, as well as being amongst the least advanced

in their class, are likely to fall increasingly further behind their average and above-average counterparts as they progress through the primary years. The key advantage of individualised instruction over group settings is the possibility of tailoring instruction precisely to the learners' needs. Of course, specialised knowledge and expertise on the part of the teacher is crucial for achieving the goal of well-tailored instruction, which is why in the cases of both Reading Recovery and Mathematics Recovery great emphasis is placed on teacher professional development. As well as being well-tailored, individualised instruction by a teacher with specialised knowledge and expertise involves, at least in our experience, highly interactive forms of mathematical communication. The teacher's instructional moves are guided not only by theory and knowledge gained from prior experience but also by moment-to-moment perceptions of the child's responses and her interpretations of those responses. Thus the teacher's facility in interpreting children's arithmetical meanings and strategies is a crucial aspect of Mathematics Recovery instruction.

Problem-centred learning versus teaching skills

The instructional approaches used in the Mathematics Recovery project come under the rubric of problem-centred or inquiry-based approaches to learning (Cobb et al., 1991) which have been used in both individualised and classroom settings. The approaches may be described as taking a holistic approach to content, rather than teaching specific skills. At the same time the approach is guided by a theoretical framework which is sufficiently specific and powerful to explain and predict the learning that is likely to result from particular instruction. However, this learning can never be predicted with certainty – children continue to surprise researchers and teachers by what they learn because of or in spite of instruction (Denvir & Brown, 1986).

Results

Results of the project pertinent to the three research questions listed above are documented below in two parts. The first part focuses on the progress of 24 Mathematics Recovery participants and 13 counterparts in terms of the model of arithmetical stages. Results in the first part are used to address research questions (a) and (b). The second part contains case studies which document the progress made by four of the participants in terms of the four underlying models, namely arithmetical stages, FNWSs, BNWSs and numeral identification. Results in the second part are used to address research question (c).

Progress in terms of arithmetical stages

The progress in terms of arithmetical stages of each of the four Mathematics Recovery participants and counterparts in each of the project schools is shown in Figure 3.1. The following points can be made:

(a) Twenty-three of the 24 Mathematics Recovery participants advanced at least one stage, 11 advanced at least two stages and two advanced three stages.

(b) Nine of the 24 participants advanced to at least Stage 4.

(c) Eight of the 13 counterparts advanced at least one stage but none advanced two stages or more.

(d) In each school where a comparison was possible, the progress of the participants notably exceeded those of the counterparts.

(e) Five of the thirteen counterparts and one of the 24 participants did not advance to Stage 3.

Figure 3.1. Progress of Maths Recovery and counterpart children in each school.

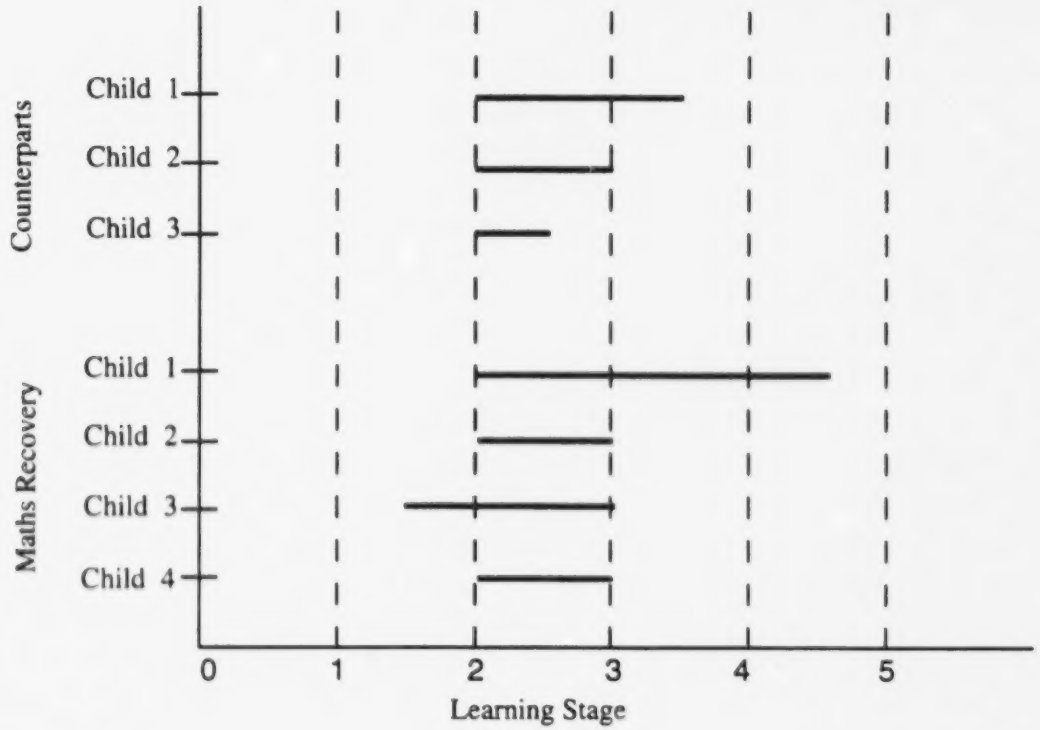

Fig. 1 School A

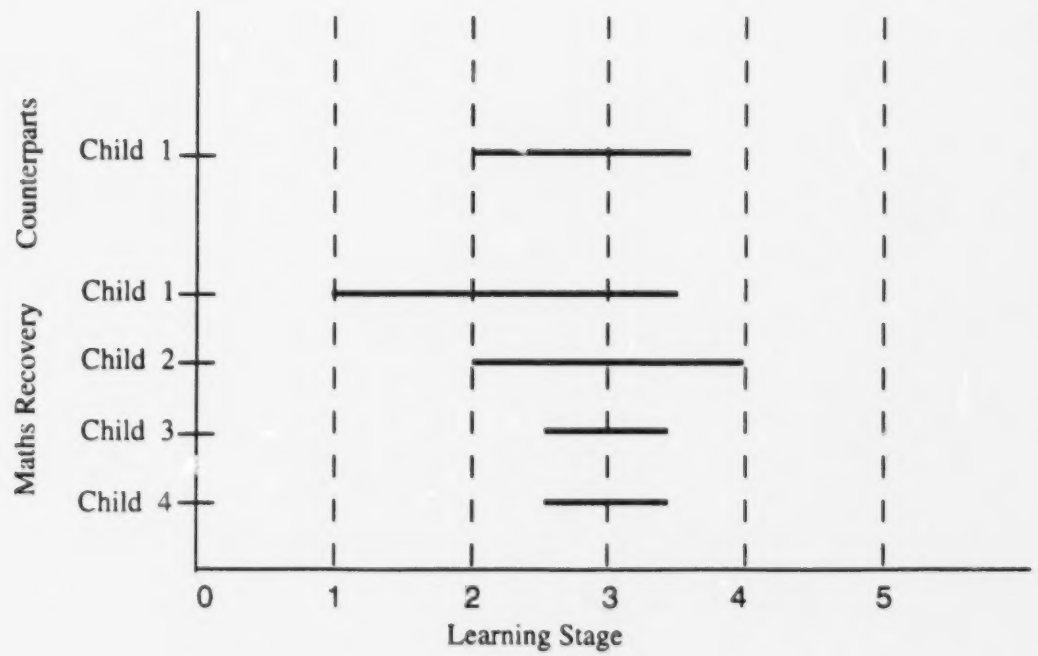

Fig. 2 School B

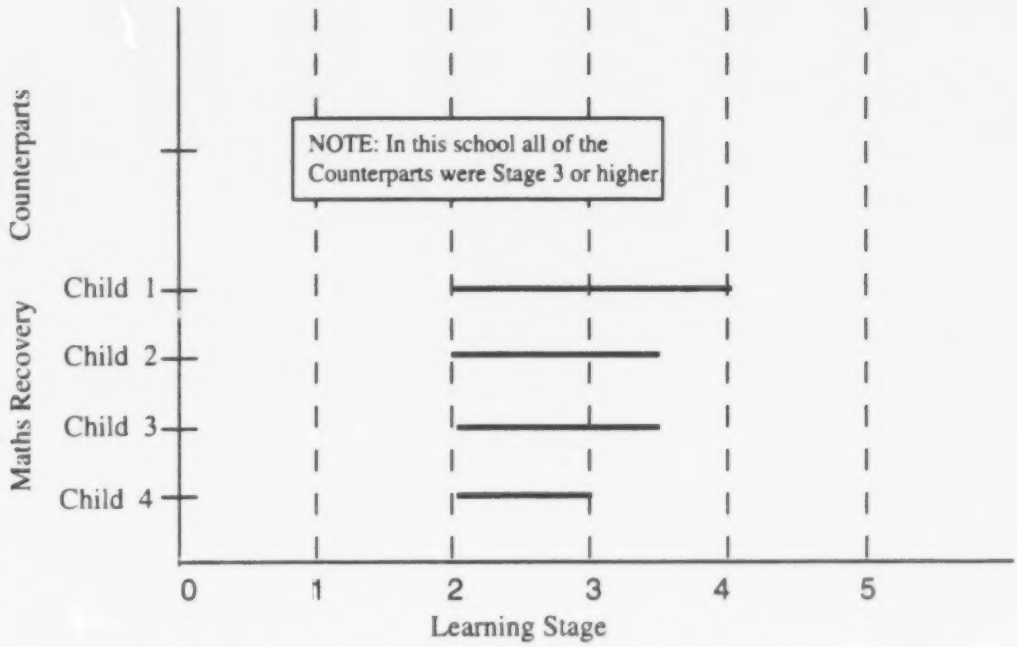

Fig. 3 School C

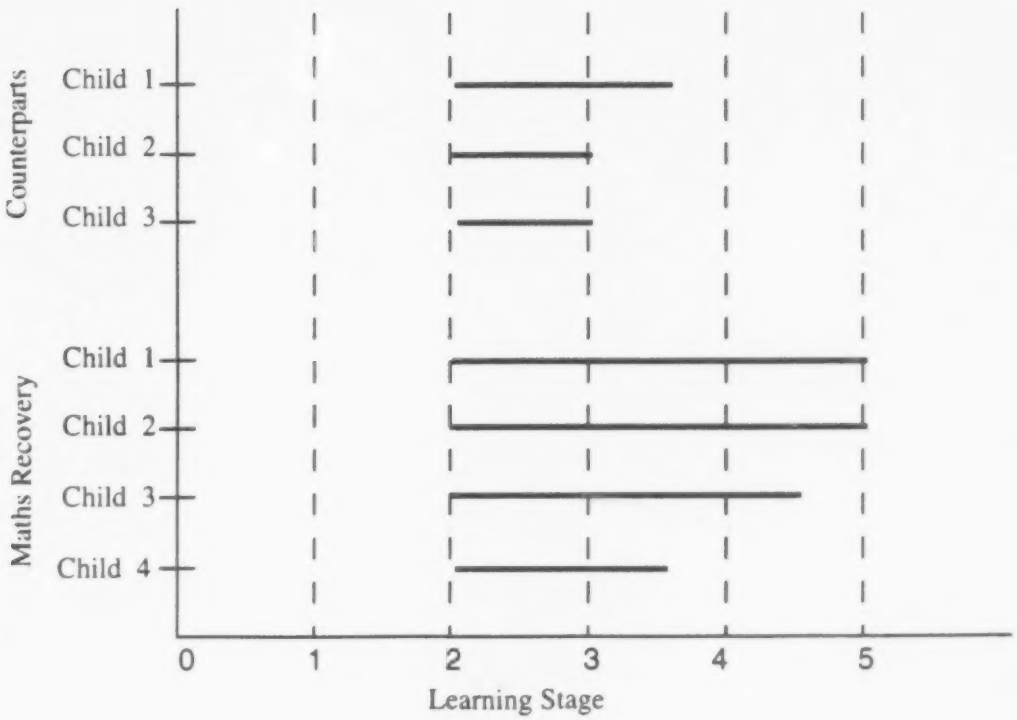

Fig. 4 School D

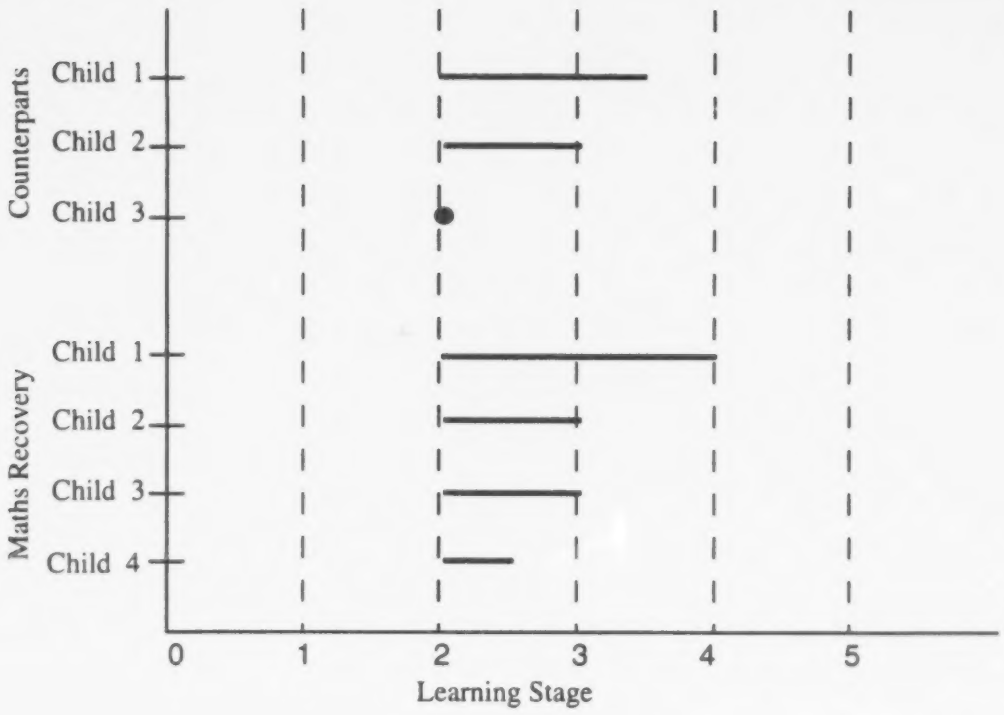

Fig. 5 School E

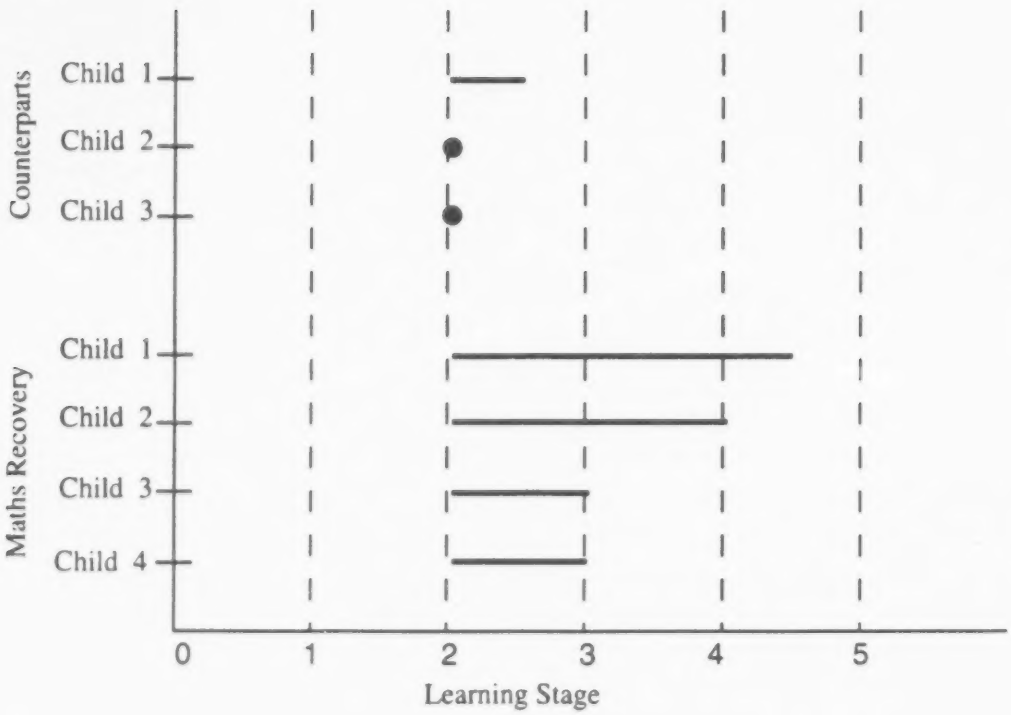

Fig. 6 School F

Case studies of progress by Mathematics Recovery participants

Case studies of the progress of four of the Mathematics Recovery participants are provided below. These case studies are illustrative of the range of participants' progress. Anonyms have been used in each case. Table 3.1 summarises their results.

Table 3.1
Progress of Case Study Students

Student	Arithmetical Stage (Max. 5)	FNWS (Max. 5)	BNWS (Max. 5)	Numeral Ident. (Max. 4)
Alice	2-3	3-5	3-5	2-*
Kevin	2-3	3-3	0-1	0-0
Patrick	2-5	3-5	3-5	3-4
Wendy	2-4	3-5	2-5	2-3

Note: Each cell shows the stage or level at the initial and final interviews. The (*) indicates a datum not available Datum not available.

The case of Alice

Alice was a student in a Grade One class at a local government school. At the time of her initial interview, she was 7 years and 6 months old. Alice left this school unexpectedly in the seventh week of the program. For this reason, her final assessments are based on teaching sessions at the end of her teaching cycle rather than a final interview, and a final assessment of her numeral identification is not available. As can be seen from Table 3.1 Alice made progress on all aspects assessed. Additionally, her classroom teacher commented on her increased confidence in classroom mathematics activities, an improvement in written tasks and a more positive self-esteem in all classroom activities. Her Mathematics Recovery teacher recorded her belief that this was attributable to experiencing success with learning activities in the Mathematics Recovery teaching sessions. She participated in 26 Mathematics Recovery teaching sessions.

In her initial interview Alice recited the FNWS from 1 to 48 with only one error, but she could not continue successfully. She produced on request the number word after given numbers in the range 1 to 20 without dropping back (that is, without reciting number words forward from one to the given number) but she needed to drop back for number words in the 20s, and was unable to find the next number beyond t30. She recited BNWSs to 1 from 10, 5 and 3, and produced on request the number word before five number words in the range 1 to 20; but she was unable to do this for numbers greater than 20. Alice identified all nine numerals asked of her in the range 1 to 20 (including 12, 15, 18 and 20). She solved three additive tasks involving two screened collections (e.g. 3 and 3, 7 and 2, 5 and 4) as well as three additive tasks involving a screened and an unscreened collection. In solving these tasks

her strategy sometimes involved counting figural unit items; on other occasions, she simultaneously raised fingers equal in number to a screened collection and then counted by sequentially touching her raised fingers to her face as she recited the appropriate number words.

Teaching frameworks designed for Alice initially involved counting forward and backward along a row of counters, using a numeral track (i.e. a row of numeral cards) with some of the numerals hidden and some visible, and reciting of FNWSs and BNWSs. Counting partially and totally screened collections to solve additive and subtractive tasks was also introduced in the first week. Alice's major progress occurred on the first day of the fourth week of her teaching cycle. Prior to this she had not been observed to count-on when solving additive or subtractive tasks, but she routinely used counting-on from this session onward. It seems likely that Alice's awareness of counting-on arose from an occasion when, after an extensive period of reflection, she realised that the sum of 20 and 3 is 23. However, there was no indication that Alice had developed concepts of tens and ones. Related to this, tasks considered likely to generate awareness of the tens and ones structure of two-digit numbers were not presented to her.

Videotapes of her teaching sessions in her final weeks indicated that Alice was now facile with FNWS to one hundred and BNWS to at least the thirties. Additionally, she could produce on request the number word after given number words up to one hundred. In her nineteenth teaching session (Week 5), she produced on request the number word before given number words in the range one to one hundred. In her twenty-second teaching session (Week 6), Alice counted-on to solve four additive tasks involving screened collections, indicating that she had reached Stage 3.

The case of Kevin

Kevin was 6 years and 10 months old at the time of his initial interview. He participated in 28 teaching sessions over the eight-week cycle. Of particular interest in the case of Kevin are (a) his initial weakness in the area of numeral identification; (b) his apparent lack of progress in numeral identification; and (c) his relative adeptness at subitising (von Glasersfeld, 1982).

In his initial interview Kevin recited on request the FNWS from 1 to 26 but said "thirty" immediately after "twenty-six". He proceeded on from 30 but could not go on successfully beyond 35. He produced the number word after each of four number words in the range 1 to 10 without dropping back, but could not do this for number words beyond 10. Kevin had virtually no success on any of the tasks concerning BNWSs. He successfully identified only two of five numerals in the range 1 to 10 and none of four numerals between 11 and 20. Kevin counted from 1 to solve additive tasks but, except for a few tasks involving very small numbers, he could not solve subtractive tasks. On three additive tasks involving two screened collections (5 and 4, 8 and 3, and 7 and 4), he first counted the larger collection from one and then continued his count to count the second collection. When counting each collection, Kevin

looked ahead and tapped out a spatial pattern on the desk as he counted. The clear indication was that Kevin was counting the items of visualised spatial patterns, figural unit items, and he seemed to be particularly adept at this. Thus it is likely that Kevin possessed a particular facility for visualising spatial patterns. Additional evidence for this was Kevin's success on subitising tasks also presented in the initial interview. Kevin successfully identified four of five regular patterns (4, 3, 2, and 5, incorrectly identifying 6 as 9) and correctly identified seven irregular patterns (3, 6, 2, 7, 4, 5, and 8). On the subitising tasks, Kevin outperformed 81 of the 94 children interviewed, including 34 children who were Stage 3 or higher.

A wide range of the instructional activities of the kinds listed earlier was presented in Kevin's teaching sessions. Documented in his teachers' evaluations and videotaped records of his teaching sessions are many instances of successful learning. Also evident are important advancements in his arithmetical strategies. At the end of Week 5, his teacher recorded that

> On Tuesday [Kevin] solved his first task with a first collection greater than 20. He solved 23 + 4 and realised himself that is was a great achievement – he looked so proud of himself.

Throughout the eight-week teaching cycle, Kevin's teacher was very much aware that he could not identify some of the numerals from 1 to 10 and she used a range of teaching techniques to address this. At the end of Week 5 she recorded as follows:

> When [Kevin] is asked to say the number word for the numerals 4 or 5 he can be confused, yet when ordering numerals 1 to 5, he does it quickly and accurately. He knows where the symbols fit in the sequence of written numerals but the number word /numeral match has not yet been established. Why?

Kevin had significant on-going difficulties with identifying numerals including some in the range from 1 to 10. He seemed to have difficulty in generating the sound image of the number word associated with a particular displayed numeral. During the course of the Mathematics Recovery project we have observed difficulties of this kind in several children. An extensive discussion of the problem is beyond the scope of this chapter.

On his final interview, Kevin produced the number word after numbers in the range 1 to 10 but with two exceptions failed to do this for number words beyond 10. He produced the BNWSs to 1 from 3, 5 and 10 but could not state the number word before numbers in the range one to ten. Kevin failed to identify two of four numerals in the range 1 to 10, and failed to identify three numerals in the range 11 to 20. He counted-on or counted-back to solve a range of additive and subtractive tasks which were presented using screened collections, including missing addend tasks (e.g. $5 + x = 9$) and missing subtrahend tasks (e.g. $6 - x = 4$).

The case of Patrick

At the time of his initial interview Patrick was seven years and three months old. As can be seen in Table 3.1, Patrick made notable progress on all four aspects of the assessment. He participated in 31 teaching sessions during the eight week teaching cycle.

In the initial interview, Patrick could recite short FNWSs up to 99 but needed to drop back when producing the number word after a given number. Similarly, he could produce with some difficulty BNWSs from numbers up to 30, but could not state on request the number word before a given number except by dropping back. He confidently identified one- and two-digit numerals but did not identify any three-digit numerals. When presented with additive tasks Patrick typically responded by guessing initially. In some cases he was able to produce a correct response on his second attempt. With one exception where Patrick appeared to count-on, his solutions involved using his fingers and counting from one. Patrick's teaching sessions were characterised by inconsistency of performance. His teacher frequently referred to a lack of concentration and difficulty with remembering problems, but remarked that he gave accurate answers at the times when his attention was sustained.

In his final interview Patrick confidently and correctly answered all of the FNWS and BNWS tasks. In doing so he gave no indications of dropping back when producing the number word after or before given number words in the range 1 to 100. Patrick correctly identified three one-digit and eight two-digit numerals which were presented to him, as well as three of four three-digit numerals. He counted-on successfully to solve all eleven additive tasks presented to him. When solving subtractive tasks he used his fingers but only in cases where the minuend was 10 or less. For tasks where the minuend was greater than 10, he chose the more efficient of counting-back-from and counting-down-to strategies. On other occasions he used strategies which are indicative of a child at Stage 5. For example, for the missing addend task $5 + x = 9$ presented using two covered collections, he explained that the second collection must contain four counters because, "if ... five and five is ten, five and four has to be nine". In explaining his solutions to a subtractive task he said "well, if five take away three is two, 15 take away three has to be 12", and in a similar vein used $5 + 4 = 9$ to explain his answer to $9 - 5$.

The case of Wendy

At the time of her initial interview, Wendy was aged 6 years 5 months. Wendy participated in 28 teaching sessions during the eight-week teaching cycle. As seen in Table 3.1, Wendy made good progress on all four aspects of numerical understanding.

In her first interview Wendy slowly and methodically recited the number words to 99. She confidently stated the number word after given numbers less than 10, but had difficulty doing so with some of numbers in the teens and could not do so for numbers greater than 20. Wendy recited three BNWSs to one in the range 1 to 10 but was unable to recite BNWSs from numbers beyond 10. Wendy could produce on request the number word before a given number in the range 1 to 10 but to do this she needed to drop back to one. She stated the

number word before each of 14, 16 and 18 but could not do so for 19 or 26. Wendy identified thirteen numerals in the range 1 to 100 but failed to identify 31 and 66. She solved two additive tasks involving a screened and an unscreened collection but was unsuccessful on three similar tasks. She also failed to solve three additive tasks involving two screened collections and with the exception of two tasks involving very small numbers, she did not solve any subtractive tasks. When attempting to solve additive tasks Wendy consistently counted from one.

In the teaching sessions of the first two weeks, Wendy was unable or unwilling to explain to her Mathematics Recovery teacher the strategies she used to solve additive and subtractive tasks. As the weeks progressed Wendy became much more able and willing to describe her strategies. This change was accompanied by an increasingly positive attitude towards her Mathematics Recovery sessions and to school in general.

In her final interview Wendy recited three FNWSs and produced on request the number word after each of ten number words in the range 1 to 100 without dropping back. She produced on request the number word before each of nine numbers in the range 1 to 100 without dropping back, and recited BNWSs from each of 3, 5 and 12 down to 1, and from 15 to 9, 20 to 10, and 30 to 20. Wendy identified all eleven one- and two-digit numerals asked of her, but failed to identify any three-digit numerals. She counted-on to solve six additive tasks and five missing addend tasks, all of which were presented using two screened collections. She also solved six subtractive tasks in which the interviewer removed and displayed items from a screened collection. She also solved missing subtrahend tasks, also presented using screened collections. However she gave no indication that she could spontaneously use Stage 5 strategies (such as using 9 + 3 to solve 9 + 4).

Concluding remarks on the case studies

The four case studies document in detail the arithmetical knowledge of the children at the time of their initial and final interviews. Evident in the case studies is that each of the four children made significant gains in their arithmetical knowledge including progress on most aspects assessed. Also indicated is that participation in Mathematics Recovery was typically a positive and intrinsically rewarding experience for children and teachers. Finally, when one considers the close accordance between the progress in children's arithmetical knowledge and the arithmetical content of the instructional activities it seems reasonable to attribute the progress to the participation in Mathematics Recovery.

Discussion of the questions addressed by the study

(a) *In terms of the theoretical model of arithmetical stages, what progress was made by participating children over the eight-week teaching cycle?*

The progress of the Mathematics Recovery participants in terms of the model of arithmetical stages are set out in Figure 3.1, has already been noted. It can be seen that all but one of the participants advanced to Stage 3, the stage of the Initial Number Sequence, and this can be regarded as a major achievement of the program. That some children were able to advance two or three stages while others advanced one stage only can be attributed in large part to the differences among Stage 2 children. What is clear from current and earlier investigations is that children's progression to the Initial Number Sequence (i.e. Stage 3) and the stage of Part/whole Operations (i.e. Stage 5) require profound reorganisation of thinking and it cannot be assumed that progress will necessarily be achieved as a result of participation in even the best-designed teaching program. The most one can do as a teacher is to present instructional activities that, in terms of underlying theoretical models, have a good likelihood of leading to a reorganisation of thinking. Important in this regard are opportunities for children to construct arithmetical strategies and reflect on the results of their strategies.

(b) *How does the progress of participating children compare with that of the non-participating counterparts who, at the beginning of the teaching cycle, were at Stage 2 or below?*

Consideration of the progress of the counterparts indicates that amongst children who may be considered to be the least advanced, many make important progressions in their arithmetical knowledge with no more than regular classroom instruction. At the same time these advancements are unlikely to be sufficient to bridge the ever widening gap between the least advanced and the average learners (Wright, 1994). Further, a minority of the least advanced made virtually no progress during large periods of the school year. A compounding effect of this lack of progress is that, by the end of their second year of school, these children are less advanced than a significant percentage – estimated at 16% earlier in this report – of the most advanced children beginning the first year of school.

One can hypothesise a link between the lack of progress just referred to and particular instructional settings. For example, if the instructional activities that the child encounters on a daily basis require no more than perceptual counting (that is, counting collections of visible items from one), then the child may fail to develop strategies characteristic of the more advanced stages. A very common beginning strategy used by children in additive or subtractive situations is what we call "counting forward from one, three times, using visible items". For example, 14 - 9 involves counting out 14 items, then counting 9 of these and removing them, and finally counting those remaining to obtain the answer 5. Our belief is that extensive use of instructional settings where primitive strategies of this kind flourish can be

counterproductive to the reorganisations of thinking which are so important to advancement of arithmetical knowledge.

(c) *In terms of the four models of young children's numerical development what progress can be made in a Mathematics Recovery program?*

The progress documented in the case studies are typical of the Mathematics Recovery participants. Like Alice, Patrick and Wendy, the vast majority of children advanced across all aspects of numerical understanding. Thus extension of the range of FNWSs and BNWSs in which the child is facile, typically from 1 to 10 at the commencement of the teaching cycle to 1 to 100 at the end, was accompanied by the development of counting-on and other more sophisticated arithmetical strategies, and coming to know the names of two- or three-digit numerals. Kevin's ongoing difficulty in learning the names of numerals in spite of intensive efforts by an otherwise very successful Mathematics Recovery teacher was a sober reminder that, even in very favourable instructional situations, successful learning is not guaranteed. In hindsight one can speculate that a necessary ingredient for addressing Kevin's difficulty in numeral identification is a much richer understanding of children's strategies for generating sound images of number names. Currently this is not available to researchers or teachers.

Conclusion

The work of Steffe and colleagues (Steffe et al., 1983; Steffe & Cobb, 1988) indicates that prenumerical (i.e. below Stage 3) six-year-olds may require up to two years to progress to Stage 5 and even then attainment of Stage 5 is not guaranteed (Steffe & Wiegel, 1993). In the case of the Reading Recovery program children are usually taught 5 days per week for 16 to 20 weeks – at least 80 lessons in all. In this study, children were taught 4 times per week for 8 weeks – a maximum of 32 lessons. Additionally, the project teachers in this study were undergoing an experimental professional development program of significantly less duration than that of Reading Recovery – six months as opposed to twelve. Considered in this light, it seems reasonable to interpret these results as a positive indication of the viability of the Mathematics Recovery program. The continuing progress of the Mathematics Recovery children and the counterparts will be assessed at yearly intervals in order to evaluate longer term benefits.

Participation by teachers in the Mathematics Recovery project has had important additional benefits for classroom teaching. Teachers have found that the instructional activities from the one-to-one instructional settings of Mathematics Recovery are easily adapted for classroom use. As well, Mathematics Recovery principles and techniques are applicable not only to low-attaining children, but apply equally to average and high-attaining students. The theory and techniques of Mathematics Recovery have also been adapted for use at preschool level and levels above second grade.

Since commencement in mid-1992 we have regarded the Mathematics Recovery project as a multi-faceted research project. The focus of this chapter is but one aspect of this research – a summary and case-based report of participants' progress in the first year of school. Undertaking the project has enabled significant development and refinement of current theoretical models (Steffe et al., 1983; Steffe & Cobb, 1988; Wright, 1989), not only those relating to the stages of early arithmetical learning but also facility with number words and number word sequences, numerals and numeral sequences, and the possible roles of spatial patterns, finger patterns and materials with a quinary structure (Gravemeijer, 1994, pp. 71-73 & 182-185) in early arithmetical learning.

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Part 2

Numeration and Place Value

Chapter 4

Representations of Place Value Knowledge and Implications for Teaching Addition And Subtraction

Gillian Boulton-Lewis

Introduction

The discussion of representations and place value in this chapter is derived from a program of research, based on an information processing perspective, into children's use of representations and strategies in the first three years of school (Boulton-Lewis, 1993a, 1993b, 1993c; Boulton-Lewis & Halford, 1992; Boulton-Lewis & Tait, 1994). The intention was to assess the value and limitations of the use of representations in early mathematics learning and teaching. Representations were defined broadly to include concrete embodiments of numbers, symbols for numbers and operations, and combinations of the latter in number sentences and algorithms. The objective of the research was to identify, describe and examine critically some of the strategies and representations that children and teachers use in early mathematics. Since place value and its representation are crucial to understanding an ' learning other aspects of mathematics, place value was also investigated on its own and as it affected the learning of operations. The research concerning place value and representations is described in some detail in the body of this chapter and effects on learning of operations and implications for teaching are considered briefly in the discussion.

Representations, processing capacity and structure mapping

In mathematics, young children are presented regularly with representations in the belief that they will facilitate the learning process. Some representations are everyday materials, some

are structured or semi-structured concrete embodiments of mathematical concepts such as sticks in bundles of ten or Multibase Arithmetic Blocks, and others are representations adopted by the discipline of mathematics (e.g. number lines and symbols). All these representations are technically analogs, and can be analysed using analogy theory. According to Gentner's theory (Gentner, 1982; Gentner, 1983; Gentner & Toupin, 1986) analogies entail a structure-preserving mapping from a source to a target. In teaching mathematics a concrete representation can be thought of as the source and the concept to be taught the target. The value of a concrete representation is that it mirrors the structure of the concept and the child should be able to use the structure of the representation to construct a mental model of the concept.

It is claimed that concrete representations as analogs should reduce learning effort and serve as memory aids; provide a means of verifying the truth; increase flexibility of thinking; facilitate retrieval of information from memory; mediate transfer between tasks and situations; indirectly facilitate transition to higher levels of abstraction and can be used generatively to predict unknown facts. There are however some possible disadvantages. Mapping from a concrete representation to a concept imposes a processing load and this can interfere with the understanding of a concept; if an analog is poor or not properly understood it can generate incorrect information; and if it is not well mapped into the material to be remembered it can increase the processing or memory load of a task (Halford and Boulton-Lewis, 1992).

A child's capacity to process information, and his or her knowledge, both develop with age (Carey 1985; Chi & Ceci, 1988; Halford, 1993). The difficulty that children (and often adults) experience in cognising some concepts is due to the load imposed by the process of mapping the task into a mental model. Halford (1993) defined processing load as the amount of resources or mental effort required to perform a task at a given level. He defined capacity as the maximum resources available to an individual. Capacity varies across individuals and with maturation. Capacity overload occurs when the processing load of a task exceeds a person's available resources. Halford (1988, 1992, 1993) proposed a structure mapping theory which accounts for cognitive development in terms of increasing levels of structural complexity of tasks (hence processing load) together with capacity and knowledge. He defined four levels of mappings from one structure to another, of which the second and third, that is relational and system mappings, are relevant here. A relational mapping is one where two elements, and the relation between them, are mapped from one structure to another. For example, the relation between two unequal sets and the ">" relation between the numerals that represent them is a relational mapping. A system mapping is one where three elements, and the relation between them, are mapped into another structure. For example, addition is a binary operation which requires the union operation on sets to be mapped into the addition operation on numbers. Hence it is a concept at the system mapping level. Halford proposed that it should be possible for children to cognise addition in its most basic form from about 5 years onwards (Halford, 1988, 1992, 1993).

Place value is also a concept at the system mapping level (Boulton-Lewis & Halford, 1992), because in order to understand the part-whole relationship inherent in the place value

representations of numbers greater than 9 the child must be aware of the additive property at least, if not the multiplicative property of units regrouped as multiples of 10, 100 and so on. When children use inconsistent or non-canonical representations for numbers, as described by Miura, Chungsoon, Chi-Mei & Yukari. (1988), they probably have some knowledge of part-whole relationships between tens and units in numbers beyond 9. However they have not yet recognised that it is the convention to group by tens and then represent such groups by digits in a specified place in a number. Table 1 is a summary of the mappings at relational and system mapping levels that we propose should allow full understanding of place value.

Table 4.1

Requisite relational and system mapping knowledge to cognise place value as a binary operation

	Relational mappings
1.	Relation between number names in counting string and objects in order[a]
2.	Counting rule dependent on relation between quantity in the set and last number name said[a]
3.	Relation between written and spoken number names
	System mappings
4.	Knowledge that 12 units (for example) are equal to 10+2 units[b]
5.	Knowledge that the number twelve (for example) is conventionally represented by a 1 in the tens place and by 2 in the units place

a Cf. Gelman & Gallistel, 1978

b Cf. Steffe & Cobb's numerical composite unit as described by Bell (1990)

Place value and representations

Children in the early years of school have limited experience of numbers. Hence despite the fact that they should be able to begin to cognise binary operations it is likely that children's understanding of place value will not be consistent from one number to another, and that they will not always choose to represent a number using the most efficient materials as analogs. The latter assumption is supported by the categories for responses used by Ross (1986) and Miura et al. (1988) in research in children's development of concepts of place value and numeration. They distinguished three categories, as follows:

(a) one-to-one collection: *f* the construction used only unit blocks (e.g. 24 unit blocks for 24);

(b) canonical base ten representation: *f* the construction used the correct number of ten blocks and unit blocks, with no more than nine units in the ones position (e.g. 2 ten blocks and 4 unit blocks for 24); and

(c) non-canonical base 10 representation: *f* the construction used some other correct number of ten blocks and unit blocks that allowed more than 9 units in the ones position (e.g. 1 ten block and 14 unit blocks for 24).

Bell (1990) undertook research with kindergarten children of European and Asian background to determine the influence that number naming systems have on young children's understanding of two-digit numbers and place value. It seems "that transparent standard number word sequences such as Japanese, Chinese and Vietnamese which follow the numerical representation pattern by naming tens and units in order ("two tens three"), may be associated with distinctive place value concepts which may support sophisticated mental algorithms" (p. 1). The studies showed that where young children use a transparent number word sequence they can more readily develop concepts of *ten items as a unit* and apply this concept to operations on two digit numbers. Therefore children who have experience of transparent number systems probably have an advantage as far as knowledge of the structure of the number sequence is concerned. Cognising place value in numbers in English may present a greater problem initially because the spoken numerals in English lack reference to the tens and ones that are contained in them (e.g. eleven, twelve, thirteen, twenty, thirty). As well, numbers such as fourteen and forty are phonetically similar and the order of pronouncing the two parts of the number fourteen does not correspond with the left/right placement of the digits in the number 14.

Kamii (1986) presented an explanation for the difficulty of place value and its implications for teaching in the primary grades. She cited results of research in the United States, Canada, and Switzerland which indicated that most children in the 1st and 2nd grades do not understand that the 1 in 12, for example, means 10 and that a large number of 3rd and 4th graders do not understand place value. She described three behaviours that she maintained are not necessarily evidence of knowledge of place value. These are, first, putting out the correct number of single units for a number such as 16, or writing the correct numeral for a set of 16 objects. The numerals can be thought of as designating whole numerical quantities and do not necessarily require knowledge of the part-whole relationship of the 10 and 6 in 16. Second, research has shown that even first graders can tell that 61 is greater than 16 because they know that 61 comes after 16 in the counting or written sequence of numbers. They do not need to know that the 6 in 61 represents six tens. Third, many first graders can remember that 10 plus 10 is 20 and that 20 and 20 are 40. It is possible for them to think of ten as a unitary concept and treat 20 for example as 2 bundles, sets, ten rods or other representations of ten as a unit. They do not need to relate tens and ones. Kamii maintained, as we do, that the construction of the place value system is a problem of part-whole relationships between tens and ones.

Figure 4.1 shows the mappings that we propose must be made between the spoken word, actual objects regrouped as tens and units, and symbolic representations for a full understanding of a number such as 16. All these mappings impose a processing load and unless each is learned well enough to be automatic, then the total load will be too great and

interfere with understanding of place value. When children who do not understand place value are expected to use concrete materials as representations of numbers greater than 9 in subtraction and addition operations, the load is further increased and it is no wonder that they try to learn procedures by rote or invent "buggy" algorithms. Consequential problems for subtraction are described in Boulton-Lewis (1993a, 1993b).

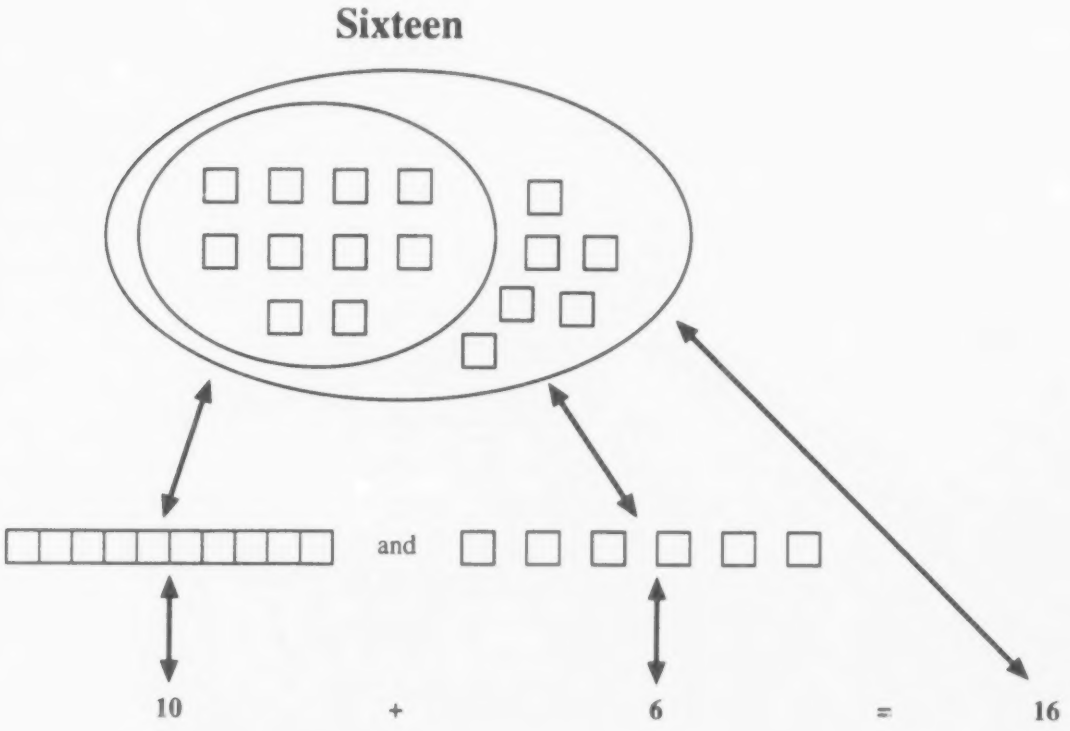

Figure 4.1: Mappings of 16, based on place value, into a 10 rod and 6 units, and then into a composition of a set of 10 and 6 units

In the study reported here we were interested in determining and analysing children's and teachers' representations and procedures for place value and considering these from the perspective of the processing loads that they would impose.

Method

The sample consisted of 28 children (10 in Grade 1, 10 in Grade 2, and 8 in Grade 3) and their teachers in a suburban Brisbane school in a low to medium socio-economic area. The mid-year mean ages were respectively 6;1, 7;1 and 8;4 years. The children were selected to represent a range of mathematical performances at each Grade level.

Each teacher was asked to identify in advance those aspects of place value that she intended to teach in subsequent weeks and the learning objectives for each child. Each child in

Grade 1 and 2 in the sample was interviewed individually before and after instruction to determine knowledge of place value and representations. The children in Grade 3 were interviewed only at the conclusion of the research to determine their knowledge of place value after three years of school. All interviews with children were videotaped.

Children were tested for knowledge of place value in 8, 10, 12, 13, 16, 18, 20, 30, 40, 50, 80, 23, 27, 34, 41, 56, 92, 108 and 150. These numbers were chosen for the following reasons: 8 could be used as a training item as well as a test of whether the child knew that there were no sets of ten in this number; 10 because it represents a single set of ten or 10 units; 12 to 18 because these are combinations of tens and units which cause difficulties for children because of the way they are spoken in the English language; 20 to 80 because they are multiples of ten with no units and because they are often confused with the teens numbers; 41 to 92 because they are combinations of tens and units of increasing size; and 108 and 150 to determine whether any of the children could represent and explain numbers beyond 100. In fact no child in Grades 1 and 2 attempted these last two numbers.

A range of materials was provided for children to use including Multibase Arithmetic Blocks (MAB) in base 10 (tens and units for Grades 1 and 2 plus some hundred blocks for Grade 3), matchsticks (or paddlepop sticks) singly and in bundles of ten, counters, a metre ruler, and the test numerals written on cards.

The child and the interviewer discussed the materials. The interviewer briefly explained any unfamiliar materials. The child was shown the first number and asked to read it; asked to put materials underneath the number to show how much it was; asked how many tens and how many ones there were in the number; and asked to explain why the number was written in that way. Each child started at a number appropriate to his or her grade level, proceeded through the numbers and stopped after difficulty with three tasks in succession.

Results

In Grade 1 the teacher expected the children to learn about tens and units in numbers to about 30. In Grade 2 they were expected to acquire knowledge of place value to about 99. Both teachers used a range of materials including single objects such as counters, Unifix cubes, MAB units, straws and sticks. The teachers sometimes joined Unifix cubes together to make rods of ten, sticks were made into bundles of ten and MAB ten rods were used occasionally. Children were asked to make numbers with materials on charts of various kinds to represent given numerals or else to put out materials and then to write the correct numerals. They also counted in tens starting at numbers such as 10 or 2 (e.g. 2, 12, 22, 92). In Grade 3 similar procedures were used for numbers into the hundreds and more use was made of MAB materials. Both Grade 1 and Grade 2 teachers said that they used a range of materials so that

the children would learn that numbers could be represented in a variety of ways, and so that the children would not become bored.

The responses of each child for the pre- and post-tests were transcribed from the videotapes and then coded for knowledge of place value, errors and representations. The coding categories were derived from the literature and the responses given by the children. The two researchers independently read a sample of responses and achieved at least 90% agreement. The categories were as follows.

Knowledge of place value

There were three categories for place value knowledge: *No knowledge*, if the child either could not answer questions correctly about the tens and units in a number or explain the value of the numerals on the basis of their place in the number; *Global knowledge* of a number (one-to-one collection as described by Miura et al. 1988), when the child gave explanations that indicated that, for example, 16 was the name for a collection of units only; and *Place value*, for responses that clearly indicated knowledge that a number consists of sets of tens and units represented according to place value.

The changes in knowledge of place value from pre- to post-test for Grades 1 and 2 are described in detail in Boulton-Lewis & Halford (1992) for the numbers (10, 12, 13, 20, 30, 23, 27) for which there were sufficient data in both Grades. The results are summarised here. The majority of explanations for these items evidenced some knowledge of place value in both Grades on the pre- and post-tests. In Grade 1 on the pre-test there were 15 responses that indicated no knowledge of place value, 16 global responses and 40 place value responses. By the post-test there were 12 responses with no knowledge of place value, 15 global responses and 43 place value responses. This indicates a minimal change from no knowledge to global or place value knowledge in Grade 1. The numbers for which less then 50% gave place value explanations in the pre-test were 12, 13 and 20 and for the post-test 12. Surprisingly, eight of the ten children gave place value responses for 27 on the pre-test and for 23 and 27 on the post-test. In Grade 2 on the pre-test eight children gave place value explanations for 10 and 12, nine gave global responses for 13, 30, 23 and 27 and all ten gave place value responses for 20. By the post-test they all gave place value responses for every number except for one child who gave a global response for 13. The results showed a steady improvement in knowledge of place value over the two years, despite some minor confusions. This was to be expected because, according to Halford (1988), children should be able to learn system mapping concepts from about five years of age. All the Grade 3 children gave place value explanations although they did not all use the most efficient materials (e.g. bundles or MAB blocks) to represent tens in all numbers.

Errors

There were three categories of errors: *Adding* of numbers, when a child responded to a number such as 13 with the answer 4; *Digits* only, when a child read a number such as 41 as 4 then 1; and *Reversal*, when a child read a number such as 13 as 30.

The number and kind of errors for reading and interpreting digits in each number are shown in Table 4.2. On the pre-test the greatest number of errors occurred for 12, 13, and 16 in both Grades 1 and 2. There were still one or two children in Grade 1 on the post-test who added both digits or who read the numbers as single digits in 10, 12, 13, and 16. Those children who attempted numbers from 20 to 92 on the post-test in both Grades did not make any of the errors that were made on the pre-test. No children in Grade 3 made any of these errors.

Table 4.2
Number and kinds of error for each item

	Pre-Test			Post-Test		
Item	Adding	Digits	Reversal	Adding	Digits	Reversal
10		1		1		
12	2	2	3	1	2	
13	2	3	3		2	
16	1	2	4		1	
18			2			
20	1	1				
30	1	1				
40			2			
50			2			
80			2			
23		1				
27		2				
34	1	1				
41	1	1				
56	1					
92	1					

Representations

Five categories were used for the different representations that children made: *One-to-one* matching of unit objects to the counting sequence; *Sequential* single units, when the digits in a number such as 21 were represented by 2 units and 1 unit; *Inconsistent*, when the representation was made up of a mixture of different objects and/or non-canonical use of sets of tens and units (for example, 2 MAB ten blocks and 13 sticks for 33); *Fingers*, for use of fingers and toes; and *Tens* for canonical use of bundles of ten or Multibase material

The most unexpected result that occurred was in the use of representations. It was expected that with practice and discussion of the use concrete representations for place value, all changes would have been from one-to-one matching or inconsistent units to tens where appropriate. However, on the items that were attempted by children in Grade 1 and Grade 2, only 50% of the sample made this change. Twenty-five percent of the sample did not change to using more efficient materials, that is, they continued to use inconsistent representations. Most unexpected of all, 25% actually changed from initially using tens to using one-to one matching of unit objects or an inconsistent mixture of materials. There was no use of fingers or toes as representations of numbers, except for 10 and 12 on the pre-test for Grade 1.

The regression is illustrated by the following protocols for one child.

Pre-test for 20

Interviewer:	What's this number?
Child:	Oh twenty, that's easy. Two tens *[Places two MAB tens under 2.]*
I:	How many tens do you have?
C:	Two tens and no ones.
I:	Why do we write the 2 here and the 0 there?
C:	Ooh because if you put that number there (0 before 2) it would make 2.

Post-test for 20

C:	This is going to be fun, I'll do all different things. Now ten *[puts out one MAB ten, then 6 counters, 2 sticks, and 1 MAB unit].*
I:	What do you have there?
C:	20 ... 1,2,3,4,5,6,7,8,9 [counts all single materials], 9.
I:	Nine, so nine and ten make twenty? Are you certain?
C:	Yes.
I:	How many tens?
C:	One.
I:	How many ones?
C:	Nine.
I:	What does 20 tell you?
C:	Twenty.

The kind of confusion illustrated above was apparently only temporary in this school. By late in Grade 3 all the children were using canonical tens representations for numbers up to 50. Two of them used inconsistent mixed representations for numbers beyond 50 and two more for numbers beyond 100.

Table 4.1 is a summary of the knowledge required, at the relational and system mapping level, to cognise place value in numbers greater than 10. The children described above who changed from using more to less efficient materials have learned that numbers can be represented in a variety of ways using ten and unit materials. In the process however they have not made strong mappings between digits in the tens place of a number, the spoken number name, and materials that most effectively represent ten. Consequently the load of processing information about such numbers, particularly in performing operations, would have been increased by the need to count, or think of all or most of the number, in units.

It is not possible to make a direct causal connection between the teachers' strategies and the children's behaviour. It seems likely, however, that the change from more to less efficient use of materials to represent place value occurred because the teachers in Grades 1 and 2 encouraged children to use a mixture of materials, rather than focusing after a while on one kind of representation and ensuring that the children could use it almost automatically.

Generally there was more success in interpreting and representing numbers such as 20, 30, 40 and even 23 and 27 than 12, 13, 16, 18. This was foreshadowed above because the spoken numbers from 11 to 19 in English do not correspond with the order in which the digits are written. The difficulty can be explained on the basis of the flowchart shown in Figure 2, which compares the items of information that the child must take into account to read, understand and represent 13 as compared with 30.

Language use

Some interesting and incorrect language was used by a few children in describing tens and units. Units or ones were described by one child as numbers "in real time" (perhaps because he had heard the term in computer jargon and confused this with the idea that these numbers were real in the sense of being single countable units as opposed to sets of ten). Another child described them as "casual numbers" (perhaps he meant that they were loose as opposed to bundles or ten rods). Whatever they meant by the words that they used, we believe that by using incorrect terms the children were increasing the difficulty of understanding that numbers are made up of tens and units.

A potentially more serious misconception was made by two Grade 2 children on pre- and post-tests. They described sets of ten as "bungles". When queried as to what she meant by "a bungle", one of the children explained that a "bungle of ten" was any collection of ten objects (not necessarily joined in any way or of the same kind), such as counters, MAB units, sticks. On the pre-test this child had used bundles of ten sticks to represent tens. Between the

two tests she had become confused about the concept of a bundle as being ten objects joined together in some way to represent a set of ten. Or else she had not understood the word "bundle" and its meaning. As stated earlier the teacher of this class believed that if she used a range of materials the children would not become bored. This probably caused the confusion. The use of the word "bungle" appears not only to be an apt mispronunciation but also to mark a move from using a canonical representation of ten to using a less efficient one-to-one representation.

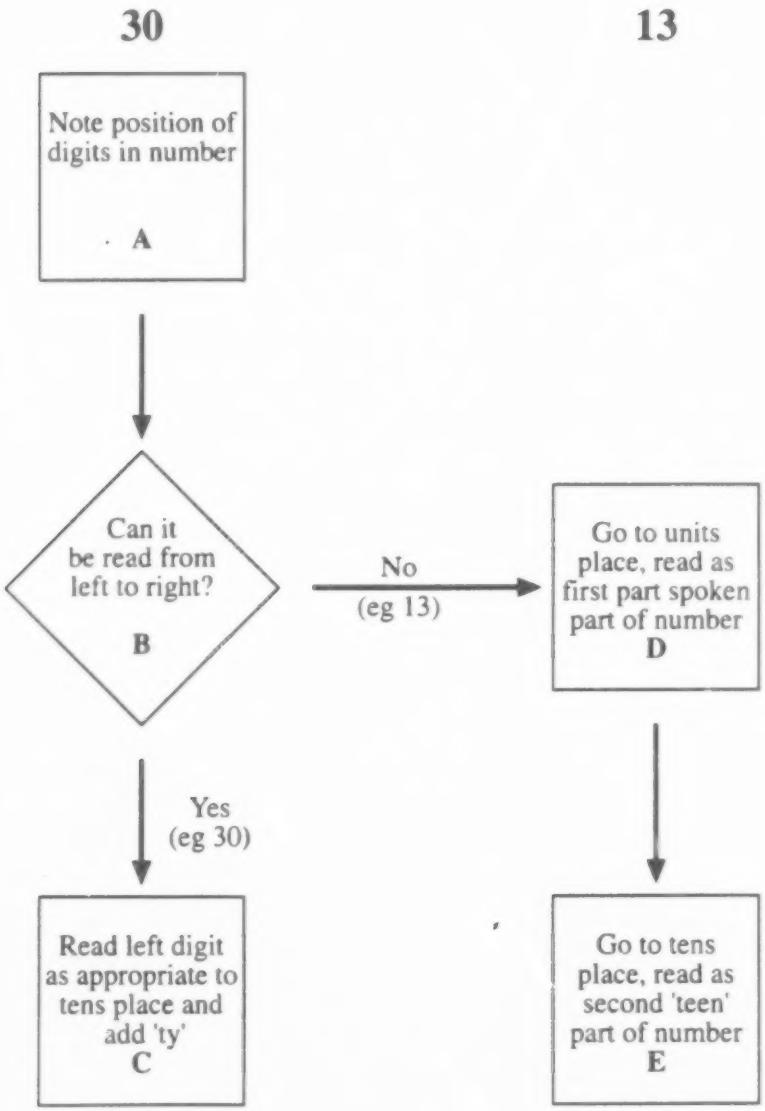

Figure 4.1: Flowchart for steps involved in reading numbers such as 30 and 13

Discussion

In summary, we found that children in our sample had some knowledge of place value even in Grade 1. They had most difficulty with 12 and 13 as predicted, and surprising success with 23 and 27. By the end of Grade 2 almost all children gave place value responses for several numbers. By Grade 3 all the children gave place value explanations. At the end of Grade 1 some children still made errors in interpreting two digit numbers by adding both digits or by reading both digits singly. None of these errors were made in Grade 2 or 3. Only 50% of children in Grade 1 and 2 changed from using less to more efficient representations of place value, 25% did not change and 25% changed to using less efficient representations.

Despite the summary above which indicates increasing knowledge of place value, of those children in Grade 1, 2 and 3 who gave place value explanations for 75% or more of the items that they attempted on the place value post-test, only 38% used this knowledge on the majority of appropriate subtraction tasks that they attempted in a parallel study (Boulton-Lewis, 1993a). Such a lack of connectedness of knowledge has been observed in other research (Lawler, 1981). Our contention is that such behaviour occurs because, despite some understanding of place value, other aspects of the subtraction task, which are not well enough known, cause a load in excess of the number of units that the child can process at once. It could also be, as Hart (1989) has suggested, that there is apparently little connection for children between a material-based approach to mathematics and formal or symbolic mathematical language, and that the challenge is to find effective transitional experiences that will help children to bridge the gap between use of concrete materials and formal written algorithmic procedures.

Boulton-Lewis also found, in the subtraction study described above (Boulton-Lewis, 1993b) and in research into addition (Boulton-Lewis & Tait, 1994), that children in grades one to three were not choosing to use concrete representations with algorithms but preferred to use mental procedures as much as possible (either with or without algorithms).

Implications for teaching

The negative change from the use of tens materials to the use of mixed materials that we observed with the Grade 1 and Grade 2 samples has implications for teachers' strategies and use of representations. The first is, that if children are to use concrete representations in operations, without increasing the processing load, they must know the material well. The second is that if the objective is to have children learn about place value then they should practice exchanging unit materials for sets of ten once they have made and understood global one-to-one responses. It is probably important for motivational reasons to allow children to use a wide range of materials to represent numbers, and it also helps them to construct mental models of numbers which map into a variety of different representations. However, they should be encouraged then to use a particular representation of sets of tens and units regularly

so that, with practice, that representation becomes well mapped into the place value of numbers.

Bell (1990) suggested that teachers of young children should consistently model two digit numbers with base 10 blocks and discuss them using terms such as "two tens and three". This seems to be a strategy worth trying but children would still need practice with mapping the concrete models and decimalised number names into numeric symbols and conventional names. Boulton-Lewis (1992, 1993c) also found a close association between place value and levels of counting and suggested that "teachers could put more emphasis on the discussion of the structure of the counting system so that children begin, as soon as they can within the limits of their processing capacity, to explain it and to understand the generative rule and how this relates to place value. This would help them to make their knowledge of the counting sequence explicit" (Boulton-Lewis, 1992, p. 17).

Fuson and Briars (1990) described success by first and second grade children with multi-digit addition and subtraction, using a learning/teaching approach which employed base-ten blocks. The positive results could be explained on the basis that the children had sufficient practice with the blocks for the mappings between the quantities and the representations to be automatic. In that case the use of blocks would not add to the processing load of performing the algorithmic procedures. Practice in mapping symbols into blocks could be achieved using games, such as those proposed by Dienes (1964) for MAB, independently of their use with algorithms.

Conclusion

The results of this research indicate that it is important for teachers to consider carefully the representations and strategies that they use in teaching place value and operations to young children. Teachers can, with the very best intentions, increase the processing load for children who are trying to cognise mathematical concepts and thus interfere with learning rather than facilitate it. An increase in processing load can occur particularly if children are encouraged to use materials that they do not know well, as concrete analogs for place value concepts in operations.

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Chapter 5

Understanding the Number System

Noel Thomas

The numeration system can be seen as a consistent and infinitely extendable system that facilitates complex mental and written notational forms of numerical representations (both whole numbers and decimal fractions). The fundamental basis of numeration is the notion of treating a group as a unit (Cobb & Wheatley, 1988; Fuson, 1988; Steffe & Cobb, 1988). Hiebert & Wearne (1992) described children's understanding of numeration as "building connections between key ideas of place value such as quantifying sets of objects by grouping by 10 and treating the groups as units... and using the structure of the written notation to capture the information about groupings" (p. 99). Understanding of place value can be difficult and slow to develop for children (Kamii, 1985; Ross, 1989). Boulton-Lewis (1994; this volume) showed that children's levels of counting were significantly related to their knowledge of, and ability to explain, place value in the first three years of school. Other studies have focused on structural aspects of numeration, identifying the child's ability to group and re-group composite units of ten and to relate this to a general structure of the base ten system (Cobb & Wheatley, 1988; Kamii, 1989).

As children make the connections between the number properties which constitute the numeration system, they develop the facility to operate efficiently with ever larger numbers. Researchers have reported on the intuitive capacity of young children in the pre-school and early school years to construct meanings for numbers (Hughes, 1986; Kamii, 1985; Ross, 1989; Steffe, 1991). As children's mathematics continues to develop through the primary school, they should acquire the mental images and connections that enable them to work with larger numbers and more complex mathematical operations in a meaningful way. However, research evidence has shown that these connections and understanding of procedures are often lacking in children's algorithms (Cobb & Wheatley, 1988; Fuson, 1990; Kamii, 1986) and it is thought that this reflects a lack of understanding of the number system. Children appear not

to transfer their initial understandings of the number system to operations with numbers and other aspects of mathematics.

There is extensive evidence (Fuson, 1986, 1992; Fuson & Briars, 1990; Wearne & Hiebert, 1988) that many children do not make the connection between the partitioning of a number into tens and ones and the decomposition of numbers in written algorithms. Using multidigit numbers meaningfully requires understanding of how to compose and decompose these multidigit numbers into multiunit structures in which numbers are thought of as collections of objects or collections of collections of objects. It is suggested (Fuson, 1990) that emphasis on two-digit number operations without trading, or delaying opportunities for interaction with numbers in the hundreds and thousands, encourages children to form misconceptions about the number system. The problem appears to be that many children base their number operations on the idea that numbers can only be added or subtracted, rather than an understanding of the number system based on multiplication.

Understanding numeration

Resnick (1983) tentatively identified three main stages in the development of two-digit numeration.

> First, there is an initial stage in which a unique partitioning into units and tens (e.g. 47 is 4 tens plus 7 units) is recognised.
> Next, in stage two, children recognise the possibility of multiple partitionings of a quantity. This second stage occurs in two phases: Multiple partitionings are (a) arrived at empirically (e.g. the equivalence of 3 tens plus 17 units to 4 tens plus 7 units is established by counting), and (b) established directly by application of exchanges that maintain equivalence of the whole (e.g. 40+7=30+17, because 1 ten can be exchanged for 10 units).
> Third, a formal arithmetic stage appears in which exchange principles are applied to written numbers to produce a rationale for algorithms (p. 127).

While Resnick discusses stages that reflect increasing formalisation of the exchange principle, both Ross (1989) and Kamii (1982) describe stages that interpret the progression of children's responses to two-digit place value questions. This can be illustrated by Ross' (1989) five developmental levels for interpreting two-digit numerals.

1. Whole numeral – the cognitive construction of a two-digit numeral is of the whole amount, no meaning is assigned to the individual digits.

2. Positional property – knowledge that the position on the right is the ones place and the position on the left is the tens place.

3. Face value – each digit is interpreted by its face value; although the tens digit may be labelled "tens", it is not recognised as a multiple of ten.

4. Construction zone – the left digit represents sets of ten objects and the right digit represents the remaining single objects, but there is unreliable task performance.

5. Understanding – individual digits represent a partitioning of the whole quantity into a tens part and a ones part; quantity of objects can be determined even for collections that have been partitioned in nonstandard ways.

The five levels that Kamii found from her study of four- to nine-year-old children was very similar to those found by Ross except that Kamii interchanged the third and fourth levels, indicating a belief that a stage of inconsistency precedes the strong reliance on face value. How does this developmental approach help us understand children's difficulties? Is it reasonable to assume that there are developmental stages through which children pass in developing understanding of the numeration system?

Thompson (1982) reported his observation of inconsistencies in the kinds of whole number, numeration-related problems that young children can and cannot solve. Many children were found to be merely learning pencil and paper algorithms with little understanding. As a result, children often appeared to know more about numeration than was actually the case - hence, the disparities observed in children's performance. Thompson developed a framework for analysing children's understanding of numeration consisting of four schema domains: reading and writing numerals; the language of number names; subitising; and numerical operations. Questions raised by Thompson were: what enables a child to come to operate with numbers that are beyond the capabilities of immediate experience; what are the conceptual requirements for having well formed concepts of ten and one hundred as numeration units; and how is it possible for children to behave so inconsistently from problem to problem? The variation in performance on apparently related tasks was explained by the suggestion that the children's knowledge of numeration is compartmentalised. Thompson (1982) also suggested that these inconsistencies might vanish when those compartments became meaningfully related. Research focusing on the connections children make between aspects of a numeration framework might be most productive.

Representing number

Close analysis of the way children construct mathematical meaning reveals a much more chaotic process than would be expected from a study of the mathematics (Hiebert, Wearne & Taber, 1991; Pirie & Kieren, 1992; Schoenfeld, Smith & Arcavi, 1993; Steffe & Cobb, 1988). Research groups focusing on children's counting strategies (Kamii, 1986; Steffe, 1991; Steffe, Cobb & Richards, 1988; Wright, 1991) and conceptual development of numeration (Bednarz & Janvier, 1988; Cobb & Wheatley, 1988; Denvir & Brown, 1986a, 1986b; Fuson, 1990; Hiebert & Wearne, 1992; Kamii, 1989; Ross, 1990) have also highlighted children's representations of the number system.

Rubin and Russell (1992) assert that children's counting, grouping, estimating and notating skills are essential elements in developing representations of the number system. They describe these elements in terms of "landmarks in the number system" (p. 136). These landmarks appear to be related to additive structure, multiplicative structure, and the generation and analysis of mathematical patterns and mathematical definitions. Rubin and Russell suggest that people who are adept with number operations e.g. computing, comparing, and estimating, have a non-uniform view of the whole number system.

Children's representations of number as some form of physical, pictorial, or notational recording have been exemplified in many studies analysing children's structural development of the numeration system (Davis, Maher & Noddings, 1990; Goldin & Herscovics, 1991; Hiebert & Wearne, 1992; Hughes, 1986; Rubin & Russell, 1992; Thomas, 1992). Thomas (1992) reported the wide variety of mental pictures of the number sequence 1 to 100 that were used by 40 children in Grades K-4. Although some aspects of structure appeared in the imagery of Grade 2 children, most Grade 4 children still did not possess the structural flexibility with number to successfully mentally manipulate two-digit numbers.

Hiebert and Carpenter (1992) describe the changes in a child's internal networks of representations as reorganisations. "Representations are rearranged; new connections are formed, and old connections may be modified or abandoned. Reorganisations are manifested both as new insights, local or global, and as temporary confusions. Ultimately, understanding increases as the reorganisations yield more richly connected, cohesive networks" (p. 69). Hence growth is portrayed as unpredictable, involving regressions as well as progressions and being, more often than not, non-linear. Pirie and Kieren (1992) described regressions in the learning of mathematics as "folding back".

There is not yet general agreement among researchers as to whether children's understanding of numeration can be described by developmental stages, and so what needs to be investigated is the relationships between the key elements of counting, grouping and place value in the development of numeration. Much of the previous work has focused upon children acquiring counting, grouping and place value skills and the additive structure of two- to four-digit numbers. Hence a cross-sectional study of children's understanding of

numeration was undertaken to investigate how the numeration system as a multiplicative structure develops. The following research questions were formulated.

1. What is the relationship between children's understanding of counting, the formulation of equivalent groups, and the base ten numeration system?
2. How is the structure of the numeration system reflected by children's representations?
3. What is the relationship between children's representations and their understanding of the formal base ten structure?

This chapter reports some preliminary results from a cross-sectional study designed to investigate these questions.

Method

Sample

A cross-sectional sample of 132 children from Grades K through 6 was randomly selected from eight State schools in the Western Region of New South Wales, Australia. This sample represented a wide range of mathematical abilities.

Classroom instruction

The teachers were surveyed to ascertain the content and materials being used in the mathematics instruction of the children selected. All participating teachers followed the official K-6 Mathematics syllabus, which emphasises the acquisition of basic facts and computational skills. A wide variety of concrete materials was used in the classroom instruction.

Task-based interviews

In order to describe individual representations of number in as much detail as possible, the study employed structured, task-based interviews based on the method used by Goldin (1993) and Davis and Maher (1993). The diverse concepts, structures and/or processes that occur inside children's minds could be inferred from the observation, classification and description of the various kinds of behaviour displayed. This method is valuable because it provides a means of investigating the interplay between the children's (internal) representations and external representations that are constructed during the interviews.

The children were interviewed and asked to solve 25 tasks. The tasks were selected after trialing in a pilot study with children from Grades K to 4. The tasks were designed to

probe understanding of numeration in the key elements of: counting; grouping/partitioning; regrouping; place value; structure of numeration; and number sense. Counting tasks probed children's counting skills. Grouping and partition tasks investigated children's construction and use of groups in both multiplication and division problem situations. Regrouping tasks focused upon the use of regrouping in number operations and place value probed knowledge and understanding of the system of numerals. Structure tasks investigated the multiplicative structure of the numeration system and number sense tasks focused upon children's sense of the way ten and powers of ten are used in the numeration system. Many of the tasks were refined from those used by previous researchers (Bednarz & Janvier, 1988; Cobb & Wheatley, 1988; Davydov, 1982; Denvir & Brown, 1986a; Labinowicz, 1985; Ross, 1986; Steffe & Cobb, 1988; Wright, 1991). The tasks were graded in level of difficulty and different subsets of tasks were given to each grade cohort. The selection of seven tasks that are reported in this chapter is provided in Table 5.1.

All interviews were conducted by the researcher in the fourth term of the school year in a room separate from the classroom. Tasks were presented verbally with concrete or visual material as indicated in the protocols in Table 5.1. The tasks were re-read to the child as often as necessary to assist him/her in understanding the intention of the question. Concrete material and pencil and paper were available on the table. The interviewer explained to the child that this material was available for use if required and that he/she was to "think aloud" as the maths activities were being done. Children were praised for their attempts, but no feedback was given as to the correctness of their responses. When a response was unclear or the child's thinking was not clear, follow up neutral questions were asked by the interviewer such as: "Can you tell me how you did that?"; "Can you describe what you did there?"; and "Did you see anything in your mind when you did that?" The length of interview ranged from 25 to 65 minutes.

Analysis of Results

The analysis of results involved classifying and coding responses for student performance and strategy use. The grouping tasks (Tasks 1, 2(i) and 2(ii)) investigated children's performance with: the use of different grouping quantities (Davydov, 1982); intuitive grouping of numbers; and the use of prepackaged (tens and ones) materials to show a representation of a two-digit number. In the abstract property of quantity task (Task 1) children needed to count two small stickers as one large sticker and so use a many-to-one coordination scheme (Watanabe, 1995). For the regrouping tasks (Tasks 2(iii), 2(iv) and 3) responses were coded for mental addition strategies related to concrete materials. For Task 3 the responses were categorised as counting, separation (collecting units of the same rank or written algorithm mentally), aggregation (one number kept whole and the other number separated into place values) or holistic strategies (decomposition and regrouping or compensation). This classification is adapted from that used by Cooper, Heirdsfield and Irons (1995).

Table 5.1

Numeration tasks: grouping, place value and structure.

Classification / purpose	Task description
Grouping: abstract property of quantity task	1. *You are collecting stickers. You can trade 2 small stickers for a large sticker. How many large stickers are worth the same as 3 large stickers and 4 small stickers?*
Grouping Regrouping - addition	Show a collection of shells packaged in bags (tens) and as loose shells (ones, only 40 provided). *I have packaged these shells into bags in order to make it easier to count them.* 2(i). *How many do you think I have put in each bag?* 2(ii). *Show me 52 shells.* 2(iii). *Use the shells to make the number which is 9 larger.* 2(iv). Children who were successful with the above task were asked if they could find another way to show 61.
Regrouping - addition (ones, tens and hundreds).	*In our lolly factory we make lollies like this one. We package groups of ten lollies into rolls like these. Then we package ten rolls into a bag like this.* 3. Show 245 represented by bags, rolls, and separate lollies. *Add to these sweets so that the number of lollies is 10 more. How many lollies have we got?* *Add to these sweets so that the number of lollies is 98 more. How many lollies have we got now?*
Structure - groupings of groupings Using structure Structure - groupings of groupings, extending the system Using structure of the numeration system	4(i). *When we have lots of bags we want to package them into boxes so that it will be easy to count them. How many bags would you put in a box?* *When we have lots of boxes we want to package them into cases so they can be transported to the shops. How many boxes would you put in a case?* 4(ii). Agree on 10 bags in a box and 10 boxes in a case. *How many lollies are there in a box?* 4(iii). *If we had 3 lollies, 5 rolls, 7 bags, 1 box and 4 cases, how many lollies would there be altogether? Show this written.* 4(iv). *If we had a case of lollies and sold 1468, how many lollies would be left?*
Structure Groupings of groupings Determine pertinence of groupings.	5(i). Show a picture of 144 marks randomly drawn. *Can you guess how many marks there are drawn there?* *I am going to do the same thing later with a friend who will be here after you. Could you do something so that, when I show him/her the sheet, he/she will be able to tell me very quickly how many marks there are? What would you do?* 5(ii). Show the same picture but with groups of ten encircled with a black line and the group of ten tens encircled in red. *Look what someone did for you* (grouping of groupings is shown). *What do you think of it? Can you now count how many marks there are?* 5(iii). If the red circle is not mentioned then the following questions are asked. *Can you explain what the red circle does? Can you now count a quicker way?*
Use of groupings of groupings in counting Assess use of groupings of tens, hundreds and thousands when quantifying large collections.	6. Show an array of 100*100 dots, drawn with spaces dividing the array into a hundred 10*10 squares. *Can you tell me how many dots are here?* *What is the quickest way of finding the number of dots?*
Structure: imagery of numbers	7. *Close your eyes. I want you to imagine the numbers from 1 to 100. Can you see a picture of these numbers?* *Open your eyes. Draw a picture of what you saw.*

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The structure tasks (Tasks 4, 5, 6 and 7) investigated imagery, use of the hundred square, groupings of groupings, and the multiplicative structure and extension of the numeration system respectively. The Lolly Factory tasks (Task 4), which were based on "The Candy Factory" third grade teaching experiment (Cobb, Yackel & Wood, 1992), investigated the way ten and powers of ten are used in extending the numeration system. The pertinence of groupings and a grouping of groupings were analysed in Task 5 (Bednarz & Janvier, 1988). For Task 6, children's responses to quantifying the dots in the array were coded according to the strategies of recognising the pattern of hundreds and multiplying by one hundred, multiplying a hundred by a hundred through looking at the whole array, and recognising the pattern of hundreds and then thousands within the array. For the visualisation task (Task 7) the external representations of the counting sequence 1-100 task were categorised and coded according to three dimensions: (a) the type of imagistic representation identified by the pictorial, ikonic and notational recordings; (b) the level of creative structural development of the number system; and (c) evidence of a static or dynamic nature of the image.

Results

Instead of presenting the results for each task separately, they will be grouped together and examined for what they say about the various aspects of the children's understanding of the numeration system.

Recognition of Place Value Structure

Figure 1 shows children's performance on the various tasks which tested their recognition of place value structure. These tasks simply asked children to recognise or represent a number or to use place value structure in counting. No calculation was involved.

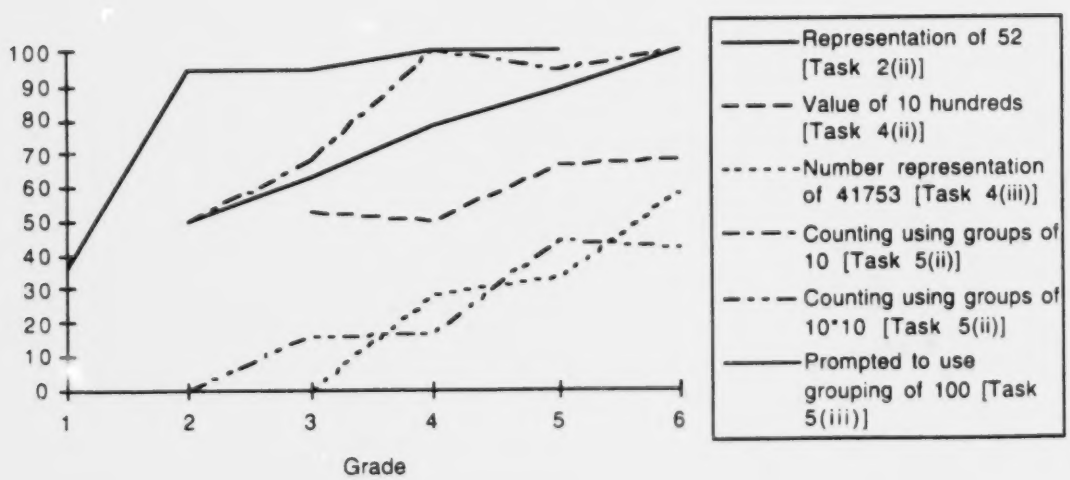

Figure 5.1: Percentage of sample correctly performing place value recognition tasks

By the end of Grade 2, most children could represent the 52 shells using the pregrouped material [Tasks 2(ii)]. A surprisingly large number (increasing from 50% at Grade 3 to 68% at Grade 6) recognised that a box of lollies (containing 10 bags of 10 rolls of 10 lollies) held 1000 lollies [Task 4(ii)]. The number who could recognise the number of lollies in a collection when they were also packed in cases of 10 boxes [Task 4(iii)] grew steadily from none in Grade 3 to 58% in Grade 6. These findings are consistent with the results of another task, not described above, in which children were asked to identify the value of individual digits in given numbers. Children showed high competence up to the thousands place (68%, 94%, 100% and 100% in Grades 3 to 6 respectively), but at Grade 6 there were still 42% who could not identify the ten thousands place. It would appear that most children master place value in numbers with up to 4 digits by Grade 3, but that progress after that is much slower.

Performance on the other two tasks suggests a modification to the above conclusion. Given a picture of 144 marks grouped in tens (circled in black) with ten groups of ten circled in red, the number of children in Grades 2 and 3 who used the tens grouping to count the marks [Task 5(ii)] was considerably less than the number who used the tens grouping in Task 2(ii); Tasks 5(ii) did not become nearly as easy as Task 2(ii) until Grade 4. The number of children who recognised and used the fact that the red circle enclosed 100 marks [also part of Task 5(ii)] was much smaller, and consistently smaller than the number of students who recognised the structure of 1000 [Task 4(ii)]. When prompted to consider the red circle [Task 5(iii)] the number of children who used the enclosure of 100 marks was consistently below the number of those who used tens groupings in Task 2(ii) or Task 5(ii). (Unfortunately, there was no task in which students were simply asked to represent a three-digit number; but such a task would probably be easier than Task 4(ii).) One explanation for these discrepancies is that students have learned to interpret certain concrete materials (bags, blocks, bundles, etc.) representing the number system but have not reached the general level of understanding needed to interpret unfamiliar groupings (circled marks) in the same way.

Calculations drawing on place value knowledge

Figure 5.2 reports the performance of three calculation tasks where students could or must have used their knowledge of place value.

By Grade 3 over 90% of the children were successful in adding 9 to the grouped collection of 52 shells [Tasks 2(iii)]. However, up to Grade 4 the majority counted on 9 and even in Grade 5 only 67% added 10 and subtracted 1 or otherwise made use of the base ten structure.

Many strategies were used for adding 98 to 245 in Task 3, but most children counted on in tens and then ones (counting strategy) or separated one or both numbers into places values and added each part separately (separation and aggregation strategies). The number of students who took advantage of the base ten structure to find more efficient methods (holistic strategies) increased from 11% in Grade 3 to 32% in Grade 6. These children (especially those in Grade

3, who had had no instruction in the subtracting three-digit numbers) were probably using methods they had invented themselves.

Very few students (none in Grades 3 to 5) were able to find 10000 - 1498 [Task 4(iv)], although many more had apparently been able to recognise the value of a five-digit numeral [Task 4(iii)].

These results suggest that most students learn to follow various procedures which are based on the place value system, but have not understood the system deeply enough to invent alternative methods when appropriate or to deal with larger numbers outside their common experience.

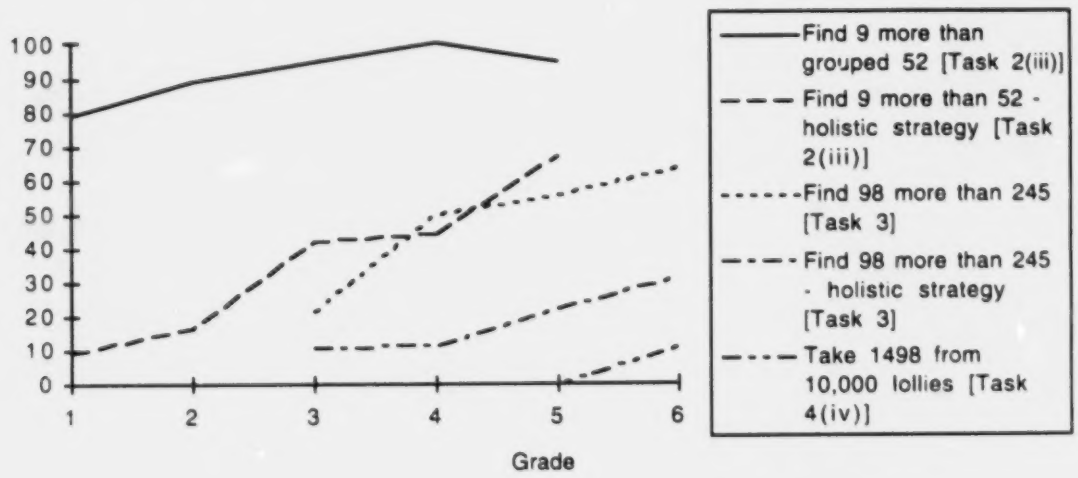

Figure 5.2: Percentage of sample correctly performing place value calculation tasks

Significance of recursive grouping by tens

Several tasks sought to find if children would spontaneously group by tens recursively in order to "make counting easier". These tasks are to be distinguished from the tasks discussed so far, where a grouping by tens was given by the interviewer. The results are shown in Figure 3.

The numbers suggesting grouping by tens for counting shells [Task 2(i)] or marks [Task 5(i)] show a gradual increase from about 20% in Grade 2 to about 60% in Grade 6. A range of grouping numbers were considered appropriate. Most of those who suggested grouping by tens could not offer a reason for their choice. A notable exception was Andrew (Grade 1), who responded, "About ten in each ... then we only have to put 10 tens to make

100". Given the use in schools of a variety of concrete materials to model the place value system - all of which are, of course, based on grouping in tens - this is indeed a surprising result. It suggests that many students are not aware, even at the most basic level, of the purpose or usefulness of our place value system.

The numbers suggesting recursive grouping by tens for packing lollies [Task 4(i)] or counting marks [Task 5(i)] was even smaller. It is again surprising that so few students are aware of a further fundamental characteristic of our numeration system.

Results from two other tasks would seem to confirm that many students are unable to use the structure of the numeration system effectively. By Grade 2, most students' drawings of the numbers 1-100 [Task 7] had developed either a linear or array structure, but there was little change in the mix of visualisations from then until Grade 6. Only 32% of the drawings in Grades 2-6 showed an array structure. In another task, not described above, children were asked to use a hundred square to find the number to be added to 84 to make 100. The number of children who were able to find the answer without counting on in ones increased from 32% in Grade 3 to 79% in Grade 6.

If the array structure of the numbers 1-100 is poorly known, it is not surprising that students have difficulty with larger numbers where grouping by tens is recursively repeated.

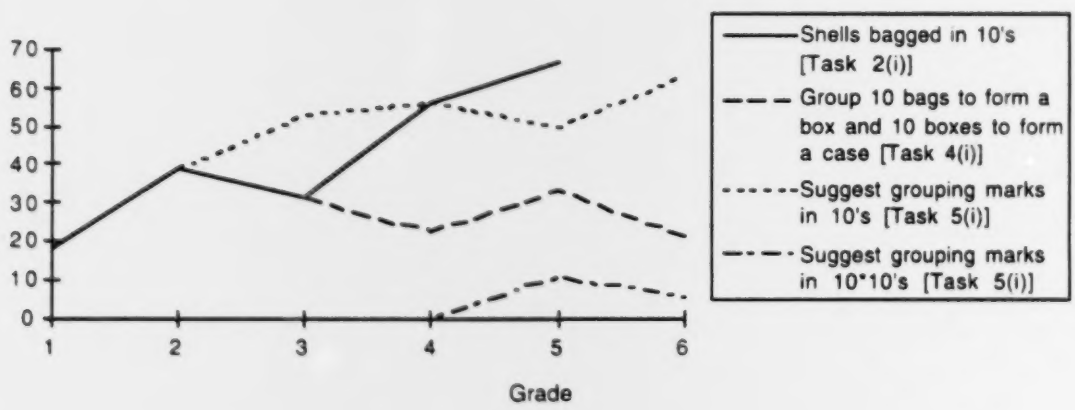

Figure 5.3: Percentage of sample suggesting grouping by tens in various tasks

Understanding the meaning of multiplication

The action of grouping by tens, so basic to the place value system, is closely associated with the operation of multiplication. Recursive grouping by tens is linked with repeated multiplication or exponentiation. The results of three tasks related to students' understanding of multiplication are shown in Figure 5.4.

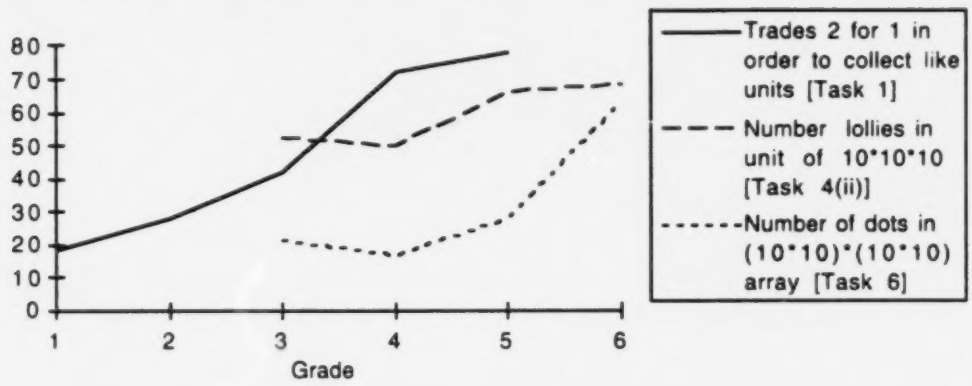

Figure 5.4: Percentage of sample correctly performing multiplication tasks.

Children showed an increasing success at calculating the result of trading two stickers for one [Task 1] - a multiplicative task with the semantic structure of ratio (see Mulligan & Mitchelmore, this volume). This task is similar to that of relating the values of successive places in a numeral, and the shape of the developmental curve resembles that of several of the place value tasks shown in Figures 1-3.

The task of calculating the number of lollies in 10 bags of 10 rolls of 10 lollies [Task 4(ii)] has already been mentioned (see Figure 1): There is relatively little improvement between Grade 3 and Grade 6. Task 6 involved a further recursion: The pattern could be regarded as made up of 10 rows of 10 groups of 10 rows of 10 dots. Successful students invented several strategies. For example, after they had determined that there were 100 dots in each square, some counting by 100s, 100 times; some counted by 100s to find that there were 1000 in the first row of squares and then counted in 1000s, 10 times; and some determined that there were 100 squares and multiplied 100 by 100. Most of the children in Grades 4 to 6 (89%, 89% and 100% respectively) recognised the pattern of 100s but many (72%, 61% and 37%) could not complete the calculation, that is, they were unable to cope with the recursion. In these three grades, about one third of the successful students used the most sophisticated strategy of multiplying 100 by 100.

Performance on Task 6 vividly illustrates the difficulties children experience relating recursive grouping to repeated multiplication. It may be conjectured that children have little experience with arrays or with repeated multiplication. It is no wonder that they also have increasing difficulties coping with the place value system as the numbers get exponentially larger.

Discussion

It appears from this study that many children in Grades 1-6 are familiar with concrete materials used to represent grouping of numbers, but still rely on unitary counting. They may show good performance on 2-digit calculations, but generally use poor methods and cannot extend their success to numbers with larger numbers of digits. There is in general a weak awareness of structure and, in particular, of the multiplicative nature of this structure. Nevertheless, some children acquire a good understanding of place value and develop their own efficient strategies spontaneously, confirming many similar findings among younger children (e.g. Cooper, Heirdsfield & Irons, 1995).

The results emphasise the importance of units and multiunits (units of more than one) in understanding the structure of the numeration system. The way that children deal with the units of one and ten influences their understanding of larger numbers (Cobb & Wheatley, 1988; Steffe & Cobb, 1988). A child who uses ten as a singleton unit might be able to recite the decade numbers (i.e. skip count in tens) but makes no sense of the increments of ten–the units of one and ten co-exists but are not coordinated. Only children who can coordinate the units and various multiunits can use these units in mental strategies for operations on larger numbers.

In the present study, there were a substantial number of students in Grades 2 and 3 who were not successful in recognising and using groupings of ten to quantify a collection of objects. Bednarz and Janvier (1988) report a similar high percentage in regular Grade 3 classes. By contrast, in Grade 3 classes which had been following an experimental constructivist teaching approach for 3 years, only 4% of the children did not see the pertinence of the presented grouping. Bednarz and Janvier also report that 30% of the experimental group but none of the control group recognised and used two orders of grouping, compared to 15% in the present study.

Children will also not realise that multiunits are related and can be exchanged if they do not understand the abstract properties of quantity (Davidov, 1982), one of the conceptual underpinnings of multiplication. In a recent study, Clark and Kamii (1996) reported that although some children develop multiplicative thinking as early as Grade 2, most children still cannot demonstrate consistent thinking in Grade 5. The present study confirms these results. A substantial minority (about 20%) of the Grade 3 students had developed such an intuitive understanding of powers of ten that they could use the recursive multiplicative structure of the array of 10,000 dots to count the number of dots successfully. But by Grade 6, there were still a significant number who could not even count 10 groups of 10 groups of 10.

Implications for teaching

The findings of the present study could be a result of school experiences which focus on the numbers 1 to 1000 (the limits of the Dienes blocks) and on analytical, algorithm-related techniques using place values separately. Simply asking for the expanded form of numbers, as so often happens in class exercises, is not sufficient to develop the necessary understanding of structure. The grouping and regrouping tasks showed that many children have not constructed the system of tens out of the system of ones and then the system of hundreds out of the system of tens, and so on. Many children could not extend the structure or produce it in the structure tasks, and were unable to construct the structure of 10*10*10*10 as 100*100 or 1 000*10 - there was also some evidence that they could not partition numbers meaningfully into components based on place values. More investigation is needed of how children naturally group and partition. Ten is a natural grouping number for children but one hundred is not. It appears that children have a natural affinity for grouping in fives and tens (fingers), but when the system needs to be extended they do not readily use the recursive process of grouping groupings. Furthermore it seems that children cannot visualise the extension of the numeration system.

A problem is that, at the time that children are being taught place value of hundreds and thousands, they should be acquiring and relating multiplicative skills but the focus is usually on addition and subtraction algorithms. They may be learning multiplication and division tables but are not necessarily learning how to extend the grouping structure of the numeration system. If we have evidence that children in Grade 3 do not have an equal grouping structure for multiplication and division, then we can explain their lack of understanding of the base ten structure which requires groupings of groupings of groupings of ten. This recursive structure is far more difficult to imagine or represent than simple equivalent groups, which can be readily represented using repeated addition (Mulligan & Mitchelmore, in press). If place value is learned as just assigned values for the positions of numerals and not as a multiplicative relationship (Rubin & Russell, 1992), learning each positional value becomes mechanical.

Multiplication is also related to partitioning – the need to partition numbers into equal groups. If you are dealing with numeration you need to partition one hundred into ten groups of ten as well as seventy eight plus twelve. Confrey (1994) argues that there is a more primitive construct, which she called "splitting", which is needed for multiplication and that the recursive procedure of repeated splitting leads to the notion of exponential functions. Therefore it is asserted that the basis of developing the structure of numeration should be on multiplicative grouping structures and not only on grouping for addition and subtraction that relates to bridging tens and hundreds.

Could it be that there is too much emphasis in the early years on addition and not sufficient on multiplication (splitting, partitioning and grouping)? The focus of the work with Dienes blocks is on addition and subtraction and not on equal groupings. The problem is also that Dienes blocks are structured to easily reflect the values of ones, tens, hundreds and

thousands, and so to use them to extend the system, one needs to rename the pieces or to build extra imaginary pieces. This is not commonly done in classrooms. Students must see concrete embodiments (like Dienes blocks) and the notational system as a reflection of each other (Thompson, 1992). This raises the question, should children do addition and subtraction with Dienes blocks as well as algorithms before multiplicative groupings? Young children need to do more activities that use multiplicative structures, e.g. groupings of groupings - leading to the notion of powers. A wider range of contexts for multiplication, such as the cartesian product model and the relationship between length, area and volume, need to be provided.

Children might also be encouraged to construct images of the numeration system and to represent these images (Wheatley & Brown, 1994) as they are making connections between the key elements of counting, grouping, addition, multiplication, regrouping, place value, zero as a place holder and powers of ten. Active processing of images is required in order to develop relational understanding. As images are built up from other images the representations do not become "autonomous" until the idea makes sense and can be used itself as a tool in mental thinking. Mental images can be described, drawn, compared and discussed. As internal structures are being developed the external visualisations may not be conventional or uniform in nature from one child to the next - they will reflect each child's unique internal constructions at that time (Thomas & Mulligan, 1994a, 1994b, 1995).

Children's understanding of the elegance and power of the structure of the Hindu-Arabic numeration system must contribute greatly to their efficient use of number. An appreciation of the power of the base ten system develops over an extended time as children construct understanding of grouping, regrouping, base ten, place value and zero as a place holder, and make the connections necessary so that the system eventually becomes transparent. A child's competence with counting, using equivalent groups and regrouping is the basis of multiplicative structures, and for working with units and multiunits. Research reported in this chapter shows that many young children need more active help in developing an adequate understanding of the structure of the numeration system.

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Part 3

Fraction Knowledge

Chapter 6

Some Results and Curriculum Implications from Recent Research on Young Children's Pre-Fraction Knowledge[1]

Robert Hunting

In this chapter, results from several studies of young children's conceptions of division, sharing, and fractions are summarised. In particular, the nature and extent of the dealing strategy used by preschoolers to share items into equal portions is described and home and social factors that might be related to the cognitive skill of dealing are discussed. Also, qualitative differences in children's conceptions of one half are outlined. Finally, curriculum issues bearing on how best to provide a sound foundation for advancing children's mathematical knowledge are addressed.

Defining pre-fraction knowledge

We need first to define what is meant by *pre-fraction*. To do this we need to explain the term *fraction*. Fractions are mathematical notation and usually occur in one of two forms–common or vulgar form (for example, $\frac{1}{2}$) and decimal form (for example 0.5). Fraction symbolism is used to signify *rational numbers*, that is numbers of the form $\frac{a}{b}$, where a and b are integers (positive or negative whole numbers) with the restriction that b cannot be zero. The distinction between the terms *fraction* and *rational number* is analogous to the distinction between *numeral* and *number*, that is, between symbol and idea. Meanings for fractions are

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multi-faceted. They include concepts like parts of wholes, ratios of quantities, measurements, and implied division.

It is important to understand the distinction between a symbol and its meaning. Symbols such as numerals, words, and even graphic images have no meaning in and of themselves. Meaning is experiences or actions that have come to be associated with symbols. The challenge for teachers of young children is to determine possible meanings that might be available to children, or which could reasonably be expected to be made available through the social interactions of home and school. When I speak of pre-fraction knowledge, I am talking of social and cognitive foundations of experience that might underpin concepts together with formal notation we teach to our children as part of their cultural heritage. At root in this definition is a belief that mathematics must be meaningful for learners of all ages.

Meanings for subdivision tasks

Young children have a variety of meanings and responses to tasks involving the subdivision of physical quantities such as food and other materials. It is important to understand children's conceptions because subdividing or breaking up material is a critical source of experience for children. Historically the mathematics of fractions is based on the notion of breaking up or "fracturing" quantities. An example of a task we have used in several research studies (Hunting & Sharpley, 1988a, 1988b) is to provide a child with a length of macrame string and two or three dolls, with a request to cut the piece of string (with scissors) so that the dolls could have a skipping rope each. A related task is to ask a child to share up 12 cracker biscuits between two or three dolls so that each doll is happy and there is no fussing or fighting.

The two examples given above represent two different types of task, which in turn involve different solution strategies. The macrame string task involves subdividing a continuous quantity. The crackers task involves subdividing a discontinuous or discrete quantity. Research by us and others (Hiebert & Tonnessen, 1978; Miller, 1984) suggests that children use different kinds of mental processes to solve each task type. One kind of process is a sequential systematic procedure which we have called "dealing" – as you do when you divide up a deck of playing cards equally between the players. This approach is used successfully for tasks involving crackers. The second kind of process involves intuitive estimation and measurement strategies, where eyeballing end-points and estimating the place where the first cut will be made is of importance. Sub-division of continuous material seems to require these strategies. Here a more holistic global appraisal seems to take place. We do not know very much about mental processes involved here, but we do observe backward and forward eye and head movements, as well as hand and finger actions.

As well as the influence of task context or type on how children respond to these kinds of tasks, other important behaviours observed include the following:

- *Non-exhaustion of the whole*
  Many preschoolers are not concerned that some material is left over after their efforts to divide material are complete. Possible reasons are (1) to the child's thinking, the doll's size does not warrant more material, or (2) in the home situation, food or other commodity is often kept for a later occasion. For many children sharing does not necessarily mean "use it all up", nor does it necessarily mean that each recipient will be allocated a portion.

- *Non-conservation of quantity*
  Young children may not have developed the logical structures to realise that if nothing is added and nothing is taken away from the original quantity, then the quantity remains invariant even though its physical form may have been altered due to cutting and so on. Children we have studied do not universally believe that a quantity as given is absolute. If the situation demands it, it is reasonable to expect more material to be provided from somewhere or, alternatively, to ignore some of what has been provided.

- *Serial sharing responses*
  We have observed some children in the macrame task to cut the string into two pieces for three dolls. When asked if the dolls would be satisfied, these children indicated that the dolls could take it in turns to use the rope, rather than provide ropes for each doll to use simultaneously in time. Again we expect experience in social and home situations to have influenced this response, as often use of commodities such as more expensive toys is regulated in this way.

Social practices and the politics of commodity regulation

Pre-fraction knowledge in its genesis is inextricably bound up with the social practices and politics of regulating the use of a commodity (very often, food). Teachers need to be aware of subtle nuances involved in these social interactions. An important common feature of both serial and parallel approaches, in many cases though not all, is to achieve a goal of approximating equal access to the commodity. Much sharing is serial rather than parallel because, as one parent explained, "we only have one of a lot of toys". Serial sharing would also be expected of larger play-things such as swings, bikes, and trampolines. We have more work to do to clarify just what factors contribute to young children's ability to share material equally.

Evidence suggests that interactions with a sibling may be important for developing methods of gaining a fair share. The roles of the younger and the elder sibling in regulating their respective desires for as much material as they each can get is a complex issue to investigate. Parents seem to encourage or at least accept that older children can have more–so long as the younger child doesn't complain! But to what extent are parent's decisions

determined by the "squeakiest wheel", and hence depend on personality and dispositional traits? Alternatively, to what extent is parental influence and intervention dependent on their personal tolerance to the demands of one or more of their offspring? Which of two siblings is likely to be more persistent? Is there a certain age when the younger child begins to realise the older sibling is (1) receiving more than themself, and (2) realises that such a state of affairs is something that is within their power to change? A fundamental determinant in the politics of fair shares would seem to be discrimination of which of two quantities is larger, but this is certainly not the only factor at play.

Linked to young children's apprehension of differences between amounts is the intervention of the parent in transactions relating to sharing. Walkerdine's (1988) study of how relational terms such as "more" and "less" are produced within social practices indicated that almost all the examples of *more* appearing in transcripts of mother/child conversation collected by Tizard and Hughes (1984) formed part of practices where the regulation of consumption was the object:

> "In every case initiated by the child, she either wants more precious commodities, of which the mother sees it as her duty to limit consumption, or the child does not want to finish food which the mother sees it her duty to make the child eat" (p. 26).
> Not only that, but the mother was in a position to stop the child having more, or tell the child she (the mother) was going to do more of something–a position the child was not in. Complex relationships then would appear to exist between the dynamics of siblings, parent and child, and apprehension of equality.

Personality traits may interact in negotiations related to sharing of commodities in the early years. Parents have observed personality differences between their offspring which they cannot account for in terms of child-rearing practices. Practices of sharing and distribution need to be observed at home over extended periods. Ethnic Chinese parents participating in a study by Hunting (1991) explained differences in the types of food eaten and the way it is distributed at meal times, compared to Western custom. It is possible that a practice common across cultures is the apportionment of food to individuals from a common reservoir. Such practice is likely to be highly salient to a hungry child, and as such, noticed well! It may be here that a dealing procedure begins to take root.

Three year olds have many more opportunities to observe food and other material being shared than opportunities to personally participate in its sub-division. It is likely that the role of observation and imitation, and the significance of what it is that is *noticed*, may be critical in children's acquisition of dealing knowledge. It maybe worthwhile for teachers to set up home-like play situations involving sharing, to talk with the children about what they think is involved and why they take certain actions.

Pre-schoolers know how to deal

A significant number of pre-schoolers possess a powerful, systematic dealing algorithm. In one study (Hunting & Sharpley, 1988b) 60 percent of a sample of 206 four and five year olds were observed to successfully allocate 12 items to 3 dolls using a dealing procedure. While few of these children seemed to understand the concept of one-half (much less one-third or one-fourth), they demonstrated a systematic method for successfully distributing crackers to dolls. The method involved three nested actions. The primary action was a matching of cracker to doll. This action was repeated until each doll had a cracker. A cycle of matching cracker to doll for each doll present was the secondary action. The third element of the algorithm was repetition of the cycle until all crackers were used up. As has been noted in previous research (Hunting, 1981), while the child who uses this systematic method does not necessarily know at the end of it how many crackers each doll has received, the method guarantees that each doll receives an equal number of crackers. Such a method is an ideal action-meaning base for the mathematical language and symbolism used to represent numbers we know as fractions, particularly unit fractions such as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. Sophisticated counting knowledge is not required to produce the required shares, although counting can be used to verify solutions.

Young children "deal" in several different ways

Young children may successfully distribute a discrete quantity in several ways:

- *One to one* (one item is given to a recipient at a time, in turn).
- *Many to one* (two or three items are given to each recipient, in turn).
- *Combinations of the above* (it is common to see a child begin allocating items in lots of two or three for a cycle or more, then continue using a one to one action).
- *Non-systematically* or by trial and error.

Children who use systematic methods can commence each cycle from a different position each time. Teachers should observe carefully how young children go about apportioning items. In particular, the order in which each cycle commences should be noted. For example, a child might be observed to commence the dealing cycle from a different doll each time, as in the example of five year old Jim (Pepper & Hunting, 1995):

A, B and C might represent dolls and the dots represent crackers. The flow of action proceeds from left to right. Variation of starting position could indicate an advance in thinking,

and a readiness to progress. The reason we say this is that variation of starting position is evidence that the child is monitoring an internal cycle of "lots" – in this case, threes – rather than physical perceptual markers such as dolls.

After non-systematic dealing, young children have been observed to use a variety of methods to equalise shares, including the following:

- *Subitising*
  This technical term simply means the ability to apprehend the numerosity of small collections–up to five items–without counting.
- *Informal height comparisons*
  Stacking items with a uniform thickness, as cracker biscuits, in front of the dolls allows a child to use visual height estimates (as evidenced by eye movements from stack to stack).
- *Informal length comparisons*
  We have also observed children place items out in front of the recipient dolls in lines.
- *Counting*
  Children who are rational counters (that is, children who are able to coordinate utterances of the standard number word sequence with items to be counted in a one-to-one stable manner) check the results of their sharing actions by counting the numbers of items for each recipient.

Counting and sharing

Both counting and sharing require action on discrete elements, including the logic of one-to-one correspondence. How counting and sharing relate to one another was the focus of two studies (Pepper, 1991; Pepper & Hunting, 1995).

The first study showed that dealing competence did not relate directly to counting competence for problems requiring the distribution of discrete items from a single group of items. Children demonstrated the use of systematic dealing procedures, but no obvious use of counting or measurement skills was observed. Children classified as poor counters were able to equally divide groups of discrete items and were confident about the result. How this was achieved became the subject of a second study, which examined strategies preschool children used when counting skills, visual cues and informal measurement skills were restricted. Children exhibited alternative strategies such as using a recipient as a mental cycle marker and an adjacent recipient strategy; pauses between allocations suggested a representation of "lots" corresponding to the number of recipients. Results from the second study supported the view that dealing competence does not depend on counting skill.

The fraction one-half

The fraction one-half is special for children, compared with other fractions. Experiences of composite units of two are so common, possibly even before birth. Consider the cycle of activity followed by inactivity, day followed by night, two nipples, two arms, two hands, two legs, and so on.

Young children aged 4-6 years have various meanings for one-half (Hunting & Davis, 1991) when faced with continuous material such as a play dough sausage or a set of discrete items such as cracker biscuits. Tasks involving continuous material may be solved in one of four ways.

- *A sequence of subdivisions*
  There are two distinct ways young children perform a sequence of subdivisions. First is a sequence of subdivisions from one end to the other performed on a continuous length. Perhaps the child has observed a parent or older sibling slicing food such as carrots. Children who perform multiple subdivisions need to engage in problem contexts that provoke the establishment of alternative means of responding–such as using symmetry for producing two near-equal quantities, or consideration of the next step: how to compose the pieces into two lots. The second action sequence consists of a series of cuts each made at approximately the mid-point of a previously divided portion. Pothier and Sawada (1983) called such a sequence *algorithmic halving*. Perhaps the child has observed a parent cutting pieces of sandwich bread into halves.

- *A single subdivision with gross or distinct inequality between pieces*
  Children will sometimes say that the requirement for fairness can be satisfied by alternating ownership over time. Their experiences may well have come from social interactions between themselves and one other person–a parent, for example. But in such interactions there may have been acceptance by the other person that equality of shares was not essential.

- *Two subdivisions, with remainder*
  In these actions children apparently see no necessity to exhaust the available material, and may well have never participated in interactions in which exhaustion is a necessary or useful concept in the context of use of the term half. In this category there is some evidence of a realisation that equality is important, but it begins to operate only after a first subdivision has been made. One portion is judged to be unequal with the other after the first subdivision has been performed, so an estimate is made–we infer this since it is often covert–and a second quantity is produced that will match one of the initial pieces.

- *Single subdivision, with approximate equality and exhaustion*
  Children in this category may be on the verge of a quantitative notion of one-half. There is an intention to create equal quantities, but the means to do so may be constrained by the nature of the material. Continuous material is particularly difficult to partition into a number of parts which is not a power of two. Characteristics of

behaviour consistent with this category of meaning are eye movements darting back and forth between endpoints of the material to be subdivided as an estimate is made of the mid-point, motoric oscillation of a finger or of a cutting utensil prior to subdivision, or relocation after initial placement of finger or hand prior to subdivision. A sophisticated behaviour involves folding endpoints of the material together in order to locate the centre. A child who makes a first subdivision, checks the resulting portions by direct comparison, then proceeds to trim one or both pieces in order to equalise the quantities demonstrates an awareness of equality critical for a quantitative conception of one-half.

For discrete materials, six categories have been observed:

- *Unequal subdivision of a collection with limited potential for equality*
  Here a child will be limited until a dealing procedure matures. Social activity involving investigations of the relative merits of systematic versus non-systematic dealing procedures is called for.
- *Unequal subdivision of a collection with potential for equality*
  Availability of a systematic dealing procedure distinguishes this category from the category above, but unless there is deliberate action on the part of a teacher or parent to enable the individual to integrate the actions of sharing into two equal subcollections with the term one-half, progress will not be possible. Children in this category may not be able to check the outcome of their subdivisions using numerical processes such as counting, though they may compare each resulting quantity by informal height or length comparison.
- *Equal subdivision of a collection from a visual check or estimate*
  A child may appropriately respond to the term one-half purely by chance, or may use judgments based on a gestalt of the perceptual field. Attention to equality is operating, but more sophisticated means such as counting or systematic sharing processes may be unavailable or are not seen to be appropriate.
- *Equal subdivision of a collection of items using dealing procedure*
  An individual in this category will confidently share a collection into two equal lots using a systematic dealing procedure. Counting may be used to check whether the outcome is correct. Individuals here are on the verge of being able to predict the size of the fractional unit using whole number facts and relationships.
- *Equal subdivision using whole number knowledge to anticipate the size of the share*
  When a child is given a task where a number of items is to be shared between two people, or where a quantity is to be divided into halves, whole number knowledge can be used to anticipate the size of each share. Specifically, the two times multiplication facts, or a good working knowledge of the addition sequence by twos, would serve as numerical relationships which can be used to work out in advance the solution to a halving task. Individuals who can use whole number knowledge, and in particular

doubles and halves, to anticipate the outcomes of problems of finding one-half of collections of modest cardinality understand one-half as a quantitative unit.

The farm: An example of a teaching framework

I will now describe a framework for teaching basic fraction concepts that is based on a farm yard scenario. It developed out of some work done with teachers in 1987 (Hunting, Clarke, Lovitt, & Pepper, 1991).

The Farm is a structured numerical (discrete) setting for developing initial fraction concepts. It is appropriate for children aged 4-8 years. A numerical setting such as this has several advantages over continuous contexts (for example, subdividing a play dough sausage). First, young children are known to have effective ways of making equal shares using a systematic dealing strategy. Second, the created shares can be made precise, which is important for the development of equality. Third, the teacher can evaluate a child's progress more accurately than in the continuous case, where unequal subunits may indicate poor technique rather than an immature conception of the importance of equal parts. Fourth, the possibility of informal discussions of equivalence relations is intrinsic to the medium. (The idea of equivalence is a key fraction concept. For example, one-half, two-fourths, and six-twelfths of 12 sheep are all 6 sheep.)

The Farm consists of a physical arrangement of paddocks, yards, fences, roads, and streams which the children can draw on a large sheet of paper, and toy animals, fences and trees common in early childhood equipment. The Farm is also supported by sets of suggested questions which have been designed as discussion or action starters. The questions have been grouped into three levels. The first level centres around development of confident systematic sharing procedures leading to equal sub-groups. The second level builds on the first. Here, the teacher assists children to associate conventional mathematical language of fractions with the results of subdivisions of discrete quantities – in this case farm animals and the like. Vocabulary is limited at first to the simple unit fractions one-half, one-quarter and one-third, beginning with one-half. The second level marks the beginnings of meaningful symbol learning. Level three is an extension of level two, in which children explore various contexts and learn how fractions "work" in part-whole and whole-part settings where the size of the "wholes" vary. In this framework there is no pressure to introduce fraction symbols. Symbols can be introduced at the discretion of the teacher after the fraction words have been learned. Which phase children work at should be based upon preliminary assessment of competency related to this topic.

Materials needed

A rural mat or sand box, play animals such as ducks, chickens, cattle, sheep, pigs, horses, and so on. Other useful materials include fences for making pens, yards, or paddocks, trees, buildings, toy trucks for transporting animals. Small groups of children can design and draw their own farm layout on large sheets of paper. One group might plan a three field farm; another group a four field farm, and so on.

The following are indications of questions and activity that might take place: –

Phase 1: Consolidation of sharing processes

Here is a suggested introduction:

> The farmer has just bought a farm, but he hasn't got any animals. So one day he goes to the saleyards and buys some pigs.
> Let's load the pigs on the truck and take them home to the farm.
> The farmer has two yards and he wants each yard to have the same number of pigs. Can you put the pigs in the yards?

Allow a child to distribute the pigs, then ask:

> Is there an equal number of pigs in each yard? Why or why not? (Allow different children to respond). Who thinks there is? Who thinks not?

(These questions provoke attention to methods of checking.)

Allow children to discuss and justify their answers. Also, ask the child who put out the pigs to verbalise how he/she solved the problem. Other animals can be "bought" and partitioned into two, three, and four yards as appropriate.

Discuss and contrast the methods children use (one-to-one cyclic methods, many-to-one cyclic methods, and trial and error methods) as they arise. Encourage children to explain in their own words why a systematic method is superior.

Phase 2: Integration of language and action

Start with a problem like this:

> One day the farmer bought some chickens. Here they are on the truck. He has two chicken coops.
> Can you help me put half the chickens in one coop, and half the chickens in the other coop?

Discuss what should be done. Allow a child to distribute the chickens.

> Is that half the chickens? (Point to one coop)
> Why or why not?

Encourage individuals to explain. Invite other children to evaluate the responses given:

> (Child's name), do you think that's right?

Similar problems can be posed where animals are to be placed in three and four yards. Use the words one-third, and one-fourth (one quarter can be used interchangeably with one-fourth.)

Phase 3: Extension to reverse problems, variable unit sizes, and notation

Place a small number of animals (three, say) in a paddock and say:

> The farmer has lost some of his ducks because the fence was broken. These are the ducks left. These are one-half of all the ducks that were in the yard. How many ducks did the farmer have at the beginning?

The above type of problem can be repeated using various animals and different numbers of animals.

If children find these reverse problems difficult, act out the situation using the total number at the outset. Such problems then can be extended using one-third and one-fourth, and fifths may be introduced.

Teaching implications

Various degrees of sophistication have been found in young children's responses to sharing tasks, and tasks where basic fraction language such as "one-half" has been used. It has been suggested that problematic situations, such as the example of The Farm, might be used to engage children in discussions about the relative merits of different methods of subdividing quantities, and to introduce fraction vocabulary in association with meaningful problems. Children come to school having experiences of both continuous and discrete quantities as foundations for the development of their ideas of simple unit fractions. Children who perform a sequence of subdivisions in response to a request to find one-half of a continuous quantity need to engage in problem contexts that provoke the establishment of alternative means of responding. As before, appropriate interactions may assist in refinement of this action strategy. For example, activities could be explored where it is difficult or relatively meaningless to perform subdivisions in this manner, such as sharing a small cracker between two people. A simple social activity to enhance the possibility of equality of shares inherent in a deeper understanding of one-half is for one child to cut a food item into two pieces and to let another child choose a piece first (I cut, you choose).

Activities to promote the exhaustion aspect of in one-half could include sharing a highly desirable object, such as a food item, or sharing a task such as shifting all of a load of wood or bricks. Exhaustion can be part of a sharing task either as a highly desirable end or as a necessary part of the task.

In cases where young children make unequal halves of a collection of items by a gross judgement or by trial and error, there will be limited progress until a dealing procedure matures. Social activity involving investigations of the relative merits of systematic versus non-systematic dealing procedures is called for. It is possible that children may have developed a systematic dealing procedure for sharing out a collection of items, but such knowledge is yet to become associated with the term *one-half*. Activities such as those suggested in Phase 2 of The Farm would be beneficial.

Children who can confidently deal out items resulting in equal allocations might be assisted to consider the function counting serves after allocations are made. Despite beliefs by some children to the contrary (Davis & Pitkethly, 1990) counting is not necessary to determine whether portions are equal; the dealing procedure guarantees that, but counting will confirm how many items are in each half if the child did not already know before commencing.

Teachers should be able to involve young children in problems of sharing even if those children have not yet become rational counters. Indeed, dealing tasks may assist children's developing counting skills through opportunities to check or verify the sizes of shares. The type of task given to the children will determine the extent to which counting may be needed. Tasks such as sharing 12 biscuits between 2 dolls can be successfully completed without the use of counting skills, whereas tasks that involve larger numbers of items may encourage the

use of counting. Children using other means of comparing quantities such as subitising, may have to rely on different methods to verify shares when the number of items increases. Providing tasks which entail shares numbering greater than 6 could encourage children to rely more on explicit quantitative methods such as counting or measuring.

Task settings preventing children from scanning the relative sizes of shares could also be presented. Such settings could be achieved, for example, by placing items as they are shared, into small containers such as money boxes, where perceptual cues are denied. By dealing initially with small numbers, such as eight coins to be shared into two opaque boxes, children could be shown the coins in one box but only told the number of coins in the second box. This would encourage counters of perceptual items to point or make verbal utterances for the hidden items if they are to find the total number of coins. Such a teaching strategy could be gradually extended to include problems with larger numbers. Children encouraged to become less reliant on perceptual information and more reliant on internal methods will develop more sophisticated schemes, which in turn will allow them to extend their knowledge to deal with new, more challenging situations. Through group work teachers could also have children share groups of discrete items, with discussion centred on the advantages and disadvantages of using systematic and non-systematic sharing procedures. Fraction names could be informally introduced, beginning with one-half. The notion of division is also being dealt with, allowing for conceptual development, again before formal written notation is introduced.

Keeping an important longer term goal in sight

Imagine a group of second grade children and their class teacher sitting around a table. On the table are arrangements of small wooden objects. There are two rabbits, four bears, six roosters, eight trees, 10 peacocks, and 12 worms. Starting with the two rabbits, the teacher introduces each larger set in turn, asking a child to divide that set in half, and to write a fraction symbol for the number of objects in each part. The children offer their own interpretations for each partition, but through discussion and assistance most children, in the end, write the sequence of fractions $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$. The teacher then says: "You have told me that each of these is one-half of the objects. How can these all be halves when we have got different numbers in them?" This question brings children face to face with the possibility of developing a deeper interpretation of the fraction one-half.

Our work suggests that understanding of fractions requires at least two levels of mathematical reflection. In the example described above, for instance, children are first presented with a level of mathematical structure at which they recognise that the whole collection, of peacocks say, can be divided into two equal sub-collections, and that there are five peacocks in each of those sub-collections. This requires recognition of ground-level mathematical structures: the number property of being ten peacocks, of being two shares, of

being five peacocks, and also a relation of equality holding between the two sub-collections–neither is more than the other.

If this is all they recognise, however, then there is, from the standpoint of a mathematically more sophisticated observer, a very important shortcoming in their understanding of fractions. To understand "one-half" in a deeper sense, a child must recognise something which several different mathematical structures have in common. The child has experienced a structure in halving the rabbits ($\frac{1}{2}$), and in halving the polar bears ($\frac{2}{4}$), and so on. In each of these separate cases a mathematically experienced observer sees a common relation: one rabbit is to two just as two polar bears are to four, and just as three roosters are to six, and so on.

A child divides the polar bears into two groups which are disjoint, which together exhaust the whole, and which stand to one another in a relation of one-to-one correspondence. This constitutes a pattern which incorporates various relations. When the roosters are halved in this way, a pattern results which is in some respects different, yet which in one respect is exactly the same as that which results when the polar bears are halved. The relation between the half and the whole is exactly the same in the case of the roosters as it was in the case of the polar bears. In other words, there is a relation between the two relations. This relation is a *meta-relation* between each half and each whole. Meta-relation which constitutes the ratio or proportion of one half is so general, in fact, that it has application not only in discrete cases where we halve the number of distinct individuals but also in continuous cases where we halve some quantity like volume or duration.

We have investigated children in Year 1 and 2 using a computer-based tool called the CopyCat, and found evidence that children of this age can understand common relations between numerical relations (Hunting, Davis, & Bigelow, 1991). The CopyCat is a black box-like machine which takes a whole number quantity of discrete items as input, and, for appropriate quantities, outputs a fraction of the input number (see Figure below). This environment allows a child to observe relations between certain inputs and their corresponding outputs, with the opportunity to construct a meta-relation–a relation that is common across all particular relational instances. For example, the number one-half is seen to be the meta-relation that encapsulates instances of four in, two out; six in, three out; 10 in, five out; and so on.

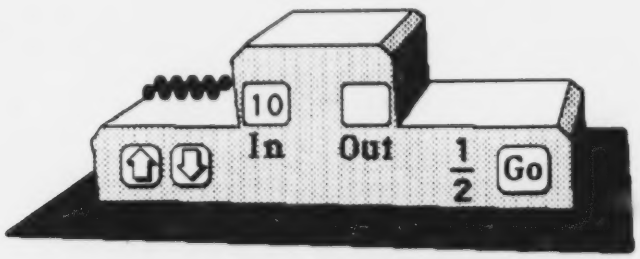

Figure 6.1

Conclusion

Research conducted with third grade children using the CopyCat (Hunting, Davis, & Pearn, in press) has suggested an inter-dependence between development of rational number knowledge and whole number knowledge. Fraction tasks in operator settings can help stimulate and extend children's whole number knowledge, and facility with whole number relationships can enable children to solve fraction comparison problems. A capacity to conceive of fractions as objects embodying numerical relationships that transcend particular instances and contexts is, we believe, a crucial goal in a person's mathematical education. Curriculum materials at all levels are frequently completely devoid of activities, problems, exercises, or projects which might develop or activate this cognitive skill. Teachers and curriculum authors need to reconsider what kinds of learning activities young children should engage in to develop their ideas about fractions. Traditionally, there has been an emphasis on continuous quantity and measurement experiences. Such an emphasis, we believe, inhibits the transition to conceptions of fractions as quantitative units and later, as meta-relations. What is needed are more dynamic environments which are action-based, and which provide more opportunities to contemplate the similarities and differences within and between phenomena. There are good grounds for supposing that problems, tasks, and explorations based on ratios and proportions would be extremely beneficial for younger children if used judiciously.

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Chapter 7

Young Children's Strategies in Solving Rational Number Tasks

Catherine Pearn

A major difficulty for all primary students, and indeed their teachers, is the topic of fractions. Kieren (1980) summarised the feelings of educators when he stated, "There has been a long-standing frustration with both teaching and learning about rational numbers" (p. 69). Southwell (1984) suggested that this difficulty could be due to the differences between the two number systems: rational numbers and whole numbers.

When confronted with rational number problems, primary school children vary considerably in both their mathematical knowledge and the strategies they use. This study focuses on the extent of whole number knowledge exhibited and the strategies used by primary school students in their solutions to rational number tasks set in various contexts.

Previous research

Over the past twenty years, research on rational number has focused on the development of basic fraction concepts, including the partitioning of a whole into fractional parts, the naming of fractional parts, and order and equivalence (see for example, Pothier & Sawada, 1983; Behr, Wachsmuth, Post, & Lesh, 1984). Kieren (1976) distinguished seven interpretations of rational number which were necessary to enable the learner to acquire sound rational number knowledge, but subsequently (1980, 1988) condensed these into five: whole-part relations, ratios, quotients, measures and operators.

Kieren suggested that difficulties experienced by children when solving rational number tasks arise because rational number ideas are sophisticated and different from natural number ideas and that children have to develop the appropriate images, actions and language to precede the formal work with fractions, decimals and rational algebraic forms. Saenz-Ludlow (1994) also maintained that students need to conceptualise fractions as quantities before being introduced to standard fractional symbolic computational algorithms. Streefland (1984) discussed the importance of students constructing their own understanding of fractions by constructing the procedures of the operations, rules and language of fractions.

Several researchers have noted how children's whole number schemes can interfere with their efforts to learn fractions (Behr, Wachsmuth, Post, & Lesh, 1984; Hunting, 1986; Streefland, 1984; Bezuk, 1988). Behr and Post (1988) indicated that children need to be competent in the four operations of whole numbers, along with an understanding of measurement, to enable them to understand rational numbers. They noted that rational numbers are the first set of numbers experienced by children that are not dependent on a counting algorithm. The required shift of thinking causes difficulty for many students.

A computer program called CopyCat was developed to help children establish basic fraction knowledge by drawing on their whole number knowledge (Davis, Hunting, & Pearn, 1993a, 1993b; Hunting, Davis, & Bigelow, 1991; Hunting, Davis, & Pearn, 1992, in press). Children who use CopyCat have been found to have a significant level of competence when considering relations between numerical quantities and expressing the abstract relation common to the numerical quantities. Hunting, Davis, and Bigelow (1991) noted that the traditional emphasis on continuous quantity and measurement experiences inhibited the development of a general concept of fractions. See also Hunting (this volume).

Saenz-Ludlow's year-long teaching experiment (1992, 1994) was undertaken to explore children's efforts to use their natural number knowledge to generate their initial conceptions of fractions. The children's conceptualisations were based on part-whole relations and underwent higher levels of abstraction over time, unhindered by lack of written symbolic notation. The use of fractional number words enhanced the abstracting process. As Saenz-Ludlow (1994) stated:

> The comparison of fractions, the inverse relation between the number of parts and the concomitant size of each part, and the inverse relation between a unit fraction and its complementary fraction were initially the result of coordinating fractional units and natural-number units (p. 82).

Pothier and Sawada (1983) and Kieren (1976) have shown that students have a rich store of informal knowledge of partitioning and equivalence of fractions. Gunderson and Gunderson (1957), and Mack (1990) concluded that students not only possess informal knowledge, which Mack defined as "applied, real-life circumstantial knowledge constructed by an individual student in response to problems posed in the context of real-life situations familiar to him or her" (p. 17), but that they are often able to connect this knowledge with formal

symbols and procedures. Mack also found that where students possessed knowledge of rote procedures they focused on symbolic manipulations. Mack's study suggested that if a strand of rational number is developed based on partitioning, using the students informal knowledge, then other strands of rational number can be developed more easily.

Steffe and Olive (1990, 1993) have shown that concepts and operations represented by children's natural language are used in their construction of fraction knowledge. The goal of their project was to identify the accommodations which children made to their counting sequences that would yield fraction schemes. Two distinct fraction schemes emerged from the research. In the iterative scheme, children established a unit fraction as part of a continuous but segmented unit. From this, children developed their own fraction knowledge by iterating unit fractions. The foundation of the measurement scheme occurred when the children's number sequence was modified to form a connected number sequence.

Lamon (1993) noted that children's modelling and counting strategies demonstrated conceptually based understanding and an adept ability to use whole number strategies. When they combined counting and matching strategies they were able to solve ratio and proportion tasks. To determine those whole number strategies that might facilitate or impede children's progress towards rational number related concepts, we would need to determine the extent of the children's whole number knowledge and the type and quality of the strategies used.

Counting plays a central role in the development of all number concepts (Fuson & Hall, 1983; Gray & Tall, 1994). Steffe, von Glasersfeld, Richards, and Cobb (1983) identified five counting types according to the type of unit the child is able to construct and the elaboration of the number sequence, and emphasised the importance of determining the strategies used by children when they encountered novel problems, for example, when counting from 10 by 3's:

> We consider important both the ability to produce the numbered word sequence in reverse and the ability to begin at any point, because these additional abilities increase flexibility in the use of the number word sequence and this is matched by an increased ease of production (p. 24).

Dubinsky (1991) defined the cognitive process of forming a conceptual entity from a dynamic process as encapsulation. Gray and Tall (1994) noted that the procedural aspects of mathematics focus on the routine manipulation of objects. When the cognitive representation of the mathematical operation and the procedure are combined, the process becomes encapsulated as a concept. Counting is encapsulated as the concept of number, repeated counting becomes addition, while repeated addition becomes multiplication. Gray and Tall (1994) proposed that proceptual thinking is a combination of conceptual thinking and procedural thinking and includes the meaningful use of known facts and procedures to solve problems, while procedural counters are dependent on the physical counting procedure.

Background and purpose of the study

To investigate the extent to which children's thinking processes when solving fraction tasks might be associated with qualitative differences in their whole number knowledge, 28 Grade 3 children of average age 8 years 2 months were interviewed in March 1992, using the clinical method (Hunting, 1983; Opper, 1977). The children were all the Grade 3 children in a predominantly middle class state-funded primary school in Melbourne's eastern suburbs.
The interviews incorporated ten tasks designed to ascertain the children's understanding of both whole number and rational number concepts. Whole number tasks included counting composite units, verbal counting, and counting arrays. Rational number tasks included partitioning tasks, fraction tasks and a ratio task.

Children were interviewed over four sessions. In the first interview session the partitioning tasks were administered; the second session involved the various fraction tasks. The ratio tasks were given in the third session and the final session was devoted to the administration of the whole number tasks. The children were individually interviewed and their responses videotaped for later analysis. The pace of the interviews was controlled by the investigator according to the child's responses. Children were not given problems that were obviously beyond their capacity. Sessions varied in length from ten to thirty minutes.

The present paper focuses on children's solutions to the rational number tasks.

Rational number tasks

Rational number tasks included various partitioning tasks, fraction tasks and a ratio task. The tasks were designed to investigate aspects of children's knowledge of partitions and fractional units, with emphasis on tasks that allowed children to demonstrate their solution strategies.

While traditional approaches to fraction teaching and learning have involved continuous quantities, recent research (Davis, Hunting and Pearn, 1993a, 1993b; Hunting, 1983, 1986; Hunting and Davis, 1991; Hunting and Korbosky, 1986) has indicated that posing tasks in discrete contexts is critical for establishing deeper relational knowledge of fractions; so most tasks were set in discrete rather than continuous contexts. The partitioning tasks were designed to test the children's ability to deal, while the fraction tasks sought to determine whether children could understand part-whole and whole-part relationships within a familiar context.

Partitioning tasks

For the Partitioning A task, each child was required to share a specified number of counters between a given number of dolls. Initially each child was given 12 counters and asked

to share these among 4 dolls so that each doll received the same number. If successful, the task was repeated using other numbers of dolls and larger numbers of counters.

The Partitioning B task involved the partitioning of a continuous quantity into a specific number of pieces. Each child was presented with 19 icy-pole sticks and asked to share these equally between 3 dolls so that each doll received an equal share. The interview focused on the strategies the child used to resolve the problem of the remaining stick.
The Partitioning C task required each child to decide how many dolls would be needed to share 20 counters equally. Encouragement to find all six ways of partitioning was given with the prompt, "Are there any other ways to share the counters, or do you think you've found them all?" The focus of this task was on the child's ability to construct multiple partitions.

Fraction tasks

Three fraction tasks were used: a unit square task, a money task and an Easter Egg task. The unit square task was adapted from a task used by Saenz-Ludlow (1992, 1994). It used a large square made up of square and rectangular pieces of various colours (see Figure 7.1) and was referred to as a square pattern or a decorated cookie. Children were first asked the number of pieces of each colour required to reproduce the whole square. They were then asked to identify various fractional parts of the square.

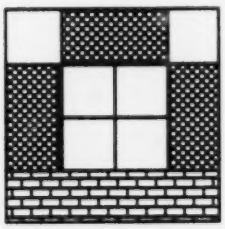

Figure 7.1: The Unit Square.

In the money task five coins were placed on the table: $1.00, 50c, 20c, 10c, and 5c. Children were asked to name the coins and then to indicate how many of each coin were needed to make one dollar. They were then asked to indicate the coins that were one half, one fifth, one tenth, and one quarter of a dollar.

Small, brightly coloured, foil wrapped chocolate Easter eggs were used in this topical task. Fractions involved were one half, one third and three fifths. For example, ten small Easter eggs were placed on the table and the child asked to indicate one-half of the eggs. The fractions one half and one third were chosen so the children's understanding of these fractions could be compared. The fraction three fifths was included to provide some indication of the degree of generalisation of fraction knowledge.

Ratio task

Four dolls were initially placed on the table in front of each child, who was asked to predict how many counters would be needed so that each doll received one counter. If they were unable to predict the number required they were allowed to share out the counters and count them. This procedure was then repeated using different numbers of dolls and counters, and included tasks where dolls shared counters.

Solution strategies for rational number tasks

The rational number tasks were analysed to determine both the knowledge and the strategies children used.

Partitioning tasks

Two kinds of strategies were observed or inferred from the behaviour of the children when solving Partitioning A tasks. These were:

(a) internal mental strategies, where answers were based on known whole number facts: and

(b) physical experiments, where solutions were constructed using available counters.

Mental strategies were inferred when:

- Children gave an immediate verbal response, or immediately counted out the correct number of counters. Eight children used automatic recall of 3 x 4 = 12 to share 12 counters between 4 dolls.
- Children derived answers using their whole number knowledge. For example, when sharing 56 counters between 4 dolls, Shannon immediately placed 14 counters in front of one doll. When questioned by the interviewer, he said, "I took 10 out and that left 16, which is 4 times 4."

Physical experiments using available counters to construct solutions included:

- Unsystematic sharing following an initial conservative estimate. For example, when sharing 56 counters between 4 dolls, Sukey started by giving each doll 7 counters, followed by 3 counters each, then 4 counters each.
- Systematic sharing by twos. For example, Tamie used sharing by twos as her strategy for sharing both 28 and 56 counters between 4 dolls.
- Systematic sharing by ones. For example, Alisha shared by ones when sharing both 28 and 56 counters and then checked to ensure a fair share. Although Sarah also shared by

ones there was no evidence that she made any attempt to count the counters after sharing.

Table 7.1 shows the different strategies used by children when sharing counters between 4 dolls. When sharing 12 counters, 5 children guessed that each doll would receive 4 counters, but when checking their solutions discovered their initial response was incorrect and reverted to sharing by ones. When sharing 28 and 56 daisy counters, children were more likely to share, commencing with a conservative estimate.

Table 7.1
Comparison of strategies used to share counters between four dolls

Strategy used	12 counters shared between 4 dolls	28 counters shared between 4 dolls	56 counters shared between 4 dolls
Immediate recall of number fact	11	1	0
Answer derived from number knowledge	0	0	3
Unsystematic sharing	0	14	12
Systematic sharing by twos	6	7	5
Systematic sharing by ones	10	6	2
Unsuccessful	1	0	1

One of the most successful students, Ozgur, appeared to use increasingly sophisticated strategies as the tasks increased in complexity. With 12 counters, he physically shared the counters one at a time. With 28 counters, he initially gave each doll 4 counters then shared the rest of the counters singly. With 56 counters, he predicted that the answer would be "10 + 4, 14, 16 + 4 = 4". From this, the interviewer inferred that he recognised that 56 could be split into two parts, 40 + 16, and realised that 10 x 4 = 40, which left 16. He knew that 16 + 4 = 4 so was able to add 10 and 4 together to obtain his answer of 14. His answer, and thus the process, was almost instantaneous.

The fact that there were 19 sticks in the Partitioning B task caused great consternation for most children. When sharing 19 sticks between 3 dolls, children thought the remaining stick meant they had made a mistake and started sharing again. Some children shared the sticks until they were all used, not checking the number in each share until invited to do so by the interviewer. Even when aware that there was an uneven share, six children had no idea what to do with the extra stick and wanted to dispose of the stick elsewhere, for example, in the bin. When dividing the remaining stick into three, children subdivided the stick in various ways. For example, Tamie indicated she would cut the stick into two pieces and stated that each doll would receive six and a half sticks each. Of the 17 children who divided the stick using two cuts, 4 children produced uneven pieces while 13 divided the stick evenly .

In the Partitioning C task, the least successful students used the strategy of sharing by ones and appeared unable or unwilling to find more than two partitions of 20. The more successful students used previously learnt whole number knowledge and systematically found more partitions. Table 7.2 shows the number of children successful in finding various partitions of 20. The greatest difficulty lay in identifying one group of 20 counters and 20 groups of one counter.

Table 7.2
Number of partitions of 20 identified in Partitioning C task.

Number of partitions	0	1	2	3	4	5	6
Number of children	1	3	3	5	10	5	1

Fraction tasks

The two parts of the unit square task invoked different solution strategies. Several different strategies were used to determine the number of pieces of each colour needed to reproduce the unit square.

- Children gave an immediate response that could be correct or incorrect. For example, Adam K. gave immediate responses and correctly estimated the number of pieces required to reproduce the square. However, immediate responses were all incorrect guesses. For example, he decided that five red rectangular pieces would reconstruct the unit square.
- Children counted using a mental image of the length of each piece. This was characterised by a head nodding or lips moving. For example, there was just a slight movement of Andrew's head as he decided that he needed 16 white pieces to reconstruct the square. This nodding was more obvious when he estimated the number of pink pieces required.
- Children measured using the rod as a measure. For example, Emmanuel used a white piece to decide on the total number of white pieces required to reconstruct the unit square.
- Children measured using an estimation of the size of the rod as indicated by fingers being moved above the square. For example, Tristan K successfully estimated that the square was equivalent to 16 white pieces but Michael overestimated the length of the white piece and decided 10 white piece were required to reconstruct the square.
- Children used previously determined knowledge to construct the answer. For example, Shannon appeared to mentally calculate the number of white blocks required and then gave instant answers about the number of red and pink blocks.

When children identified fractional parts of the unit square there were two main solution strategies.

- Children re-arranged the rods of the unit square and separated them into the required number of parts. These could be equal or unequal divisions. For example, Shannon first re-arranged the pieces and then separated them into two equal parts. He exchanged blocks for equivalent blocks of a different colour, and produced all possible combinations of blocks that would comprise half the unit square.
- Children waved their hands, or pointed their fingers, to indicate the position of the cut. For example, when asked to show one half, Bradley waved his hand above the square to indicate the vertical, horizontal and diagonal cuts required. When asked if there were any other ways he could show one half he indicated that a "curvy sort of line" could also indicate one-half.

The only strategies observed in solutions to the money task were those related to counting or previously learnt number facts. Several children were unable to respond.

- Children gave an immediate response which was either correct or incorrect. For example, Kane said there were 100 ten-cent coins and 99 five-cent coins in a dollar.
- Children counted by the denomination of the coin, correctly or incorrectly. For example, Emmanuel counted correctly the number of 5-cent pieces in one dollar, but Finley stated that there were nine 20-cent coins in one dollar.

Table 7.3
Number of children able to identify the number of each coin in a dollar

Coin	50 c	20 c	10 c	5 c
Number of successful children	10	18	24	17

There appeared to be no correlation between the knowledge of the number of each coin in a dollar (Table 7.3) and the knowledge of the fractional value (Table 7.4). For example, all 28 children stated there were two 50-cents in a dollar but 4 children did not appear to realise that one half of a dollar was 50-cents. Neither Tristan nor Diane made any response to this question, while Sukey thought one half of a dollar was 5 cents and Darren stated it was 70 cents.

Table 7.4
Number of children able to correctly identify the coins for fractional parts of a dollar

Fractional part of one dollar	One half	One fifth	One tenth	One quarter
Number of successful children	23	10	15	6

It appeared that most of the children did not understand the meaning of the terms: one fifth, one quarter and one tenth in the money context.
The contextual setting for the third fraction task was an Easter egg hunt. There were two types of problem for this task:

1. Constructing a part from the whole. Children were shown a number of small, brightly coloured, foil-wrapped Easter eggs, and told that these were all the eggs that had been hidden. They were asked to find the number of eggs representing one half, one third and three fifths.

2. Constructing the whole from a part. Children were shown a number of Easter eggs and told that these eggs were a given fractional part of the eggs that had been hidden. Children were asked to determine the total number of eggs that had been hidden.

Twenty-five children correctly determined one half of 10 Easter Eggs by recalling previously learnt number facts. Their explanations of immediate responses included: "Five is half of ten" or 5 + 5 = 10 or 2 x 5 = 10. Sixteen children were able to determine that 14 Easter Eggs had been hidden after being shown 7 eggs and being told that these were one half of those that had been hidden.

Six children successfully determined one third of 12. Ten children thought that one third of 12 was 3, while 7 children estimated that the answer was 6. Seven children were able to determine that 15 Easter eggs had been hidden if 5 eggs were one third of those hidden, of whom 6 used the multiplication algorithm 3 x 5 = 15.

The children's knowledge of fractions in the context of the Easter egg tasks fell into three categories: knowledge of unit fractions, knowledge of the fraction one half only; and no knowledge of fractions. Two children appeared to have no knowledge of fractions in the Easter egg context, ten exhibited the ability only to halve a given number; ten were able to halve and also find the whole given the half; while six children demonstrated knowledge of the unit fractions one half and one third. Although children had small Easter eggs in front of them, most were unwilling or unable to interact with these eggs while attempting a solution.

Ratio task

Two kinds of strategies were observed or inferred from the behaviour of the children. These were:

(a) internal mental strategies, where the child's answers were based on known whole number facts; and

(b) physical experiments, where the units were constructed directly using the available counters.

Mental strategies were inferred when:

- Children gave an immediate verbal response or immediately counted out the correct number of counters. For example, when Shannon was asked how many counters would be needed so that 6 dolls had 5 counters each, his immediate response was 30. His explanation confirmed that he had used the multiplication fact 6 x 5 = 30.
- Children derived their answers from their whole number knowledge. For example, when 6 dolls each received 5 counters, Sukey counted by 5, 6 times.

Physical experiments consisted of systematic sharing counters. For example, Nadia physically shared counters when sharing counters between 6 dolls given that two dolls each shared one counter. Table 7.5 compares the strategies used for each task.

Table 7.5
Comparison of strategies used for the ratio tasks

Ratio Task	Whole number strategy	Physical sharing	Unable to complete
4 dolls, 1 counter each	23	2	3
4 dolls, 2 counters each	25	3	0
6 dolls, 2 dolls share 1 counter	13	14	1
6 dolls, 3 dolls share 1 counter	24	2	2
6 dolls, 3 dolls share 2 counters	20	5	3
6 dolls, 2 counters each	23	3	2
6 dolls, 5 counters each	25	3	0

Comparison of strategies

Children were ranked according to a score obtained from the set of rational number tasks. The score for each task was determined using criteria that were dependent on both the knowledge displayed and the type of whole number strategies used. Strategies using number facts were scored more highly than those using unitary counting. The most successful students, ranked in order of their rational number scores: Adam (1), Alisha (2), Shannon (3), Ozgur (4), David (5) and Elliot (6). The least successful students were Sarah (28), Kane (27), Darren (26), Diane (25), Ken (24) and Tristan (23). The strategies used by these two groups of students were then compared.

Partitioning tasks

There were similarities and differences in the strategies used by the two groups of students when sharing counters and sticks between dolls and dividing an icy-pole stick. The most successful students tended to use the following strategies more often than the least successful students:

- Immediate recall of previously learnt addition and multiplication number facts.
- Previously learnt number facts to derive the solution.
- Sharing of counters commencing with a conservative estimate.
- Sharing of counters by dealing them one at a time and keeping a running tally.
- Division of an icy-pole stick into equal pieces.
- Checking of solutions without prompting from the interviewer.

If an incorrect solution was observed, the most successful students often adjusted the share to ensure equality. They also suggested alternative strategies that could have been used. When answers were instantaneous, it was difficult to ascertain whether the strategy stated was the one actually used to solve the task.

The least successful students tended to use whole number strategies which were less reliant on the recall of previously learnt number facts and included:

- Sharing, commencing with a conservative estimate.
- Dealing out counters one at a time with no apparent check made to ensure a fair share.
- Division of an icy-pole stick with no attempt at ensuring equal parts.
- Guessing with no attempt to use counters to confirm or refute their guess.

Although there were also similarities in the types of strategies used by the disparate groups, there were differences in the way these strategies were performed. For example, children from both groups dealt the counters one at a time to four dolls. However, the most successful students kept a running tally and checked the solution to ensure a fair share. The least successful students would only check the shares if requested to do so by the interviewer.

Both groups included students who shared using an initial conservative estimate but differences in the size of these estimates were highlighted when 56 counters were shared between 4 dolls. The most successful students initially dealt out either 10 or 12 counters while the least successful students used conservative estimates of 7, 6 or 2 counters. There was less guessing by the most successful students, and guesses were more often adjusted in the checking process. The least successful students were less inclined to check their solutions.

Fraction tasks

There were similarities and differences in the strategies used when sharing counters between dolls. The most successful students:

- Estimated the number of rods required to replicate the unit square using previous responses.
- Re-arranged rods in the unit square and substituted equivalent rods for each fractional part.
- Used previously learnt number facts to determine part from whole and whole from part.
- Demonstrated a good knowledge of one half and a partial understanding of one third.

The least successful students:

- Incorrectly guessed the number of rods required to replicate the square.
- Waved their hand over the top of the square to indicate fractional parts.
- Were unable to name fractional parts of one dollar.
- Demonstrated knowledge of one half of ten Easter Eggs but guessed solutions to other tasks.
- Guessed when asked to identify the fractional parts of a dollar.

Fraction tasks in all contexts caused difficulties for all children as they did not appear to understand the meaning of fraction term: one fifth, one quarter, and one tenth.

Ratio task

The most successful students gave instant responses and justified their response by referring to whole number addition or multiplication facts. Responses appeared to be instantaneous but successful students were able to refer to whole number strategies such as counting, doubling, and previously learnt number facts. The least successful students relied on counting procedures or guessed solutions to the ratio task.

Summary of strategies used on rational number tasks

For partitioning tasks the most successful children were flexible in the use number facts and when necessary were able to derive any knowledge required. The least successful children used the strategy of physically sharing out counters to find equal shares, were unsuccessful when faced with dividing the continuous length of the icy-pole stick, and did not find more than two partitions when sharing 20 counters between dolls.

Fraction tasks in all contexts caused difficulties as most children did not understand the meaning of most fractional terms. The unit square task most strongly highlighted differences between the two groups. The least successful students made random guesses as to the number of each piece needed to reconstruct the square, while the most successful students either estimated or used mental strategies. When asked about fractional parts of the unit square the least successful children waved their hands to indicate fractional parts of the square, with little success. The most successful students re-arranged the rods of the unit square and separated them into the required fractional parts, indicating an understanding of equivalence.

Children were more successful with the ratio tasks than they had been with the fraction tasks. The main difference in strategies between the two groups was the instant verbal responses given by the most successful students as compared to the procedural sharing process used by the least successful students.

The main difficulty experienced by most children was the lack of understanding and knowledge of fraction language. When this language was important to the task, the least successful students would not respond while the most successful students attempted a solution based on the context. Most children were unfamiliar with one third and one quarter.

Comparison with ranking on whole number tasks

Since there were considerable differences in the strategies children used to solve rational number tasks, it was felt it was important to compare the strategies used by these two groups in response to a set of whole number tasks. The whole number tasks and their subsequent analysis are not documented here (see Pearn, 1995) but scoring for the whole number tasks was similar to that for rational number tasks. Solutions using superior whole number strategies scored more highly than the procedural strategy of counting by ones. The six most successful students as given by their rational number scores received the following whole number rankings (in parenthesis): Adam (1), Alisha (5), Shannon (1), Ozgur (6), David (13) and Elliot (3). The six least successful students were: Sarah (28), Kane (25), Darren (26), Diane (24), Ken (23) and Tristan (22).

The ranking of scores for the two sets of tasks appears very similar. When children were ranked on both sets of tasks, that is, the scores from both rational number tasks and whole number tasks, the Spearman rank order correlation coefficient was found to be 0.86 ($p < .0001$). This indicates for these children, on this set of tasks, a very strong relation between the two rankings. Children who were successful with the rational number tasks and used more advanced strategies, were also very likely to be successful on the whole number tasks and to use more advanced strategies. Because of this high correlation, it is worth looking briefly at children's whole number strategies.

Summary of strategies used on whole number tasks

The least successful students appeared to have very few memorised number facts. They were inflexible with their thinking and frequently resorted to guessing without any expectation of a reasonable solution. Counting strategies were frequently inaccurate as they appeared to have difficulties co-ordinating nods and taps with the verbalised counting sequence. Some of the least successful students were unable to count the counters they could not see and were not perceptual counters (Steffe et al, 1983; Steffe & Cobb, 1988).

Gray and Tall (1994) conjectured that:

> ... the most salient memory that the child has of this process [counting] is the final object counted. ... because such a procedure occurs in time, it is hypothesised that any proceptual relationship between the input ... and the output ... is likely to be obscured by the lengthy counting routine used to obtain the solution. We suggest, then, that count-all is a procedure that extends the counting process and is unlikely to lead directly to an encapsulated procept (p.123).

For the least successful students, the focus was thus on the process of counting and not on the relationship between input and output numbers making it highly unlikely that number patterns would be noticed and generalised. It would appear that, in the context of the whole number tasks, the least successful children could be classified as procedural thinkers. Gray and Tall (1994) summarised procedural thinking as being:

> ... characterised by a focus on the procedure and the physical or quasi-physical aids that support it. The limiting aspect of such thinking is the more blinkered view that the child has of the symbolism: numbers are used only as concrete entities to be manipulated through a counting process. The emphasis on the procedure reduces the focus on the relationship between input and output, often leading to idiosyncratic extensions of the counting procedure that may not generalise (p. 132).

The most successful children were flexible in their mathematical thinking when presented with counting tasks and were equally successful in all three tasks of verbal counting, counting composite units, and visual arrays. They exhibited greater whole number knowledge than the least successful children and used various appropriate whole number strategies when unable to recall number facts. The most successful children appeared to demonstrate evidence of proceptual thought in their solution of the whole number tasks. As hypothesised by Gray and Tall, (1994):

> Proceptual thinking includes the use of procedures. However, it also includes the flexible facility to view symbolism either as a trigger for carrying out a procedure or as the representation of a mental object that may be decomposed,

> recomposed, and manipulated at a higher level. This ambiguous use of symbolism is at the root of powerful mathematical thinking and makes it possible to overcome the limited capacity of short-term memory. It enables a symbol to be maintained in short-term memory in a compact form for mental manipulation or to trigger a sequence of actions in time to carry out a mental process. It includes both concepts to know and processes to do (pp. 124-125).

The most successful students were not focused on the process but were able to use prior knowledge to increase the extent of their mathematical knowledge and understanding. When input numbers and their sum can be held in the mind simultaneously then the result is a meaningful, known fact that may be envisioned as a flexible combination of procept and procept to give a procept" (Gray and Tall, 1994, p. 124).

Conclusion

When solving rational number tasks, the most successful students demonstrated proceptual thought when solving both rational and whole number tasks. They were flexible in their mathematical thinking and used previously learnt whole number knowledge and procedures to construct the mathematical concepts needed to solve all tasks. By contrast, the least successful students demonstrated procedural thought when solving whole number tasks. They experienced some success with partitioning and ratio tasks but little or no success with the fraction tasks.

While proceptual thinkers constructed their own knowledge when tasks were presented in a real-life context, procedural thinkers appeared to be hindered by the procedure itself. They were unable to relate previous knowledge to the context and thus unable to construct mathematical understanding in a new context.

There appeared to be a significant lack of understanding of both the language and concept of fractions. This was particularly highlighted in the money and Easter Egg tasks, which should have been familiar contexts. The children in the study had already received three years mathematical education. Some of the proceptual thinkers seemed to be unable to identify a quarter of either the unit square or a quarter of a dollar simply because they did not understand the meaning of "one quarter".

Despite the difficulty with the language of the rational number tasks, the analysis has highlighted the vast difference in both children's mathematical knowledge and the type of whole number strategies they used when solving rational number tasks set in different contexts. Success in the solution of rational number tasks appears dependent on more than just superior whole number knowledge but also on the type of strategies used to apply this knowledge.

Implications For Teaching

This paper raises several questions about the traditional approach to teaching fractions in primary school. Most children had difficulty in understanding the language of fractions. While most were successful with tasks involving one half very few understood, or were successful, with tasks involving other unit fractions. If children are to succeed with rational number tasks, they need to develop the appropriate language to precede the formal work with fractions and this development must span many contexts. The vocabulary and notation of simple fractions, such as those used in this study, should be introduced much earlier in primary schooling.

Children who were the most successful with rational number tasks were those who could be classified as being proceptual thinkers while those who were least successful were those classified as being procedural thinkers (Gray & Tall, 1994). Teachers need to be aware of student's strategies and encourage them to develop more efficient strategies other than procedural strategies such as counting by ones. More emphasis on counting by numbers larger than one would assist in this transition. More tasks must be presented that encourage children to use the "count-on" and "count-down" strategies rather than "count-all".

Children were more successful with the ratio tasks than they had been with the fraction tasks. It may be that children relate the sharing aspect of this task to the sharing of lollies with friends. This would appear to be a good starting point for the teaching of fractions and the introduction of the appropriate language.

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Part 4

Computational Processes and Strategies

Chapter 8

Children's Mental Strategies For Addition And Subtraction Word Problems[2]

Tom Cooper, Ann Heirdsfield and Cal Irons

In the past, the focus of primary mathematics computation in primary mathematics has been the traditional pen and paper algorithm. However, current research has suggested that mental computation should play a major role in the mathematics curriculum (Cobb & Merkel, 1989; Coburn, 1989; McIntosh, 1992; Reys, 1984; Reys & Bargen, 1991; Sowder, 1990). This has been supported by curriculum development exemplified by the Curriculum and Evaluation Standards (National Council of Teachers of Mathematics, 1989), the National Statement (Australian Education Council & Curriculum Corporation, 1991), Bobis (1993), and Willis (1990); and mathematics syllabi (Ministry of Education, Victoria, 1988; Western Australia Education Department, 1989).

Mental computation has been supported for pedagogical and utilitarian reasons. A number of researchers (Beberman, 1985; Cobb & Merkel, 1989; Cockroft, 1982; Jones, 1989; Reys, 1985; Reys & Bargen, 1991; Sowder, 1990) have proposed mental computation as a valid computational method because it enables children to learn how numbers work, make decisions about procedures, create strategies and explore rich fields of mathematical experience. The Department of Education, Western Australia (1989) stressed that "skill in mental computation will increase agility with numbers generally, and will enhance problem solving activities". Reys (1984) suggested that mental computation promotes greater understanding of the structure of number and its properties, promotes creativity and independent thinking, contributes to the development of problem solving skills and forms a basis for developing

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2 This chapter reports on a longitudinal research project that was supported by Australia Research Council grants. This longitudinal study gathered data on children's addition and subtraction mental computation strategies in Years 2 to 6 and multiplication and division mental computation strategies in Years 4 to 6. Further information can be obtained by writing to Longitudinal Mental Computation Study, Centre for Mathematics and Science Education, c/- QUT Kelvin Grove, Locked Bag No 2, Red Hill, Queensland, Australia, 4059 (email - tj.cooper@qut.edu.au).

computational estimation. Strong arguments have also been raised as to the utility of mental computation. Maier (1977) found that people in non-school situations tended to do what he called "folk mathematics", using mental computation and computational estimation methods rather than school taught pen and paper algorithms. Carraher, Carraher, and Schliemann (1985) reported that there are many informal ways of performing calculations not related to school mathematics. It has been suggested that 80% of computation used in everyday life is done mentally (Clarke & Kelly, 1989).

Accompanying this focus on mental computation has been an emphasis on encouraging children to look for patterns and formulate their own efficient mental strategies. For example, Kamii, Lewis, and Jones (1991) recommended that children should be free to formulate their own mental strategies, as students understand algorithms better if they are allowed to go through a constructive process in line with their own natural ways of thinking.

This chapter reviews research on mental computation in addition and subtraction, describes trends found in a 2 year longitudinal study in mental computation strategies used for 2 and 3 digit word problems, and draws implications for effective teaching of mental addition and subtraction.

Defining mental computation

Sowder (1988, p.182) defined mental computation as "the process of carrying out arithmetic calculations without the aid of external devices". Mental arithmetic was described by Trafton (1986, p.207) as referring to the use of "non-standard algorithms for computing exact answers" without the use of pen and paper. Although he considered that the traditional pen and paper algorithms could be performed mentally, Trafton limited the term mental computation to non-traditional algorithms employed mentally. This paper will use both Trafton and Sowder's definitions, reserving the term "mental computation" for mental calculations larger than those covered by basic facts but including strategies normally considered inefficient for mental computation.

As a component of addition and subtraction knowledge, mental computation methods relate to the other components of this knowledge: the concepts of addition and subtraction; the basic facts; pen and paper algorithms; and word problems. Modern approaches to teaching mathematics are emphasising children's constructing their own knowledge of mathematics in relation to their previous mathematical knowledge of their world (Cobb, 1994; Kamii & Lewis, 1991; Steffe & Wiegel, 1992). This has led to an emphasis on real world problems as the starting point for understanding and computing arithmetic examples and on the usefulness of children understanding and completing computation exercises in relation to word problems (Cobb & Merkel, 1989; Lindquist, 1989).

However, there has been little research relating mental computation and word problems. Few researchers who focused on mental computation used word problems. Carraher, Carraher,

and Schliemann (1987) were an exception; they presented computation in word problem form in studies of children's mental computation strategies. Reys, Reys, and Hope (1993) presented computation in the form of applied problems in a study of mental computation performance in grades 2, 5 and 7, but they collected written answers and could not identify the strategy or the cause of error. Bebout (1990) also used word problems, but the purpose was to elicit symbolic representations; that is, the focus of her research was children's attempts to symbolically represent addition and subtraction word problems presented with materials. Other researchers (Beishuizen, 1993; Hope & Sherrill, 1987; Reys, Reys, Nohda, & Emori, 1995) restricted their presentation to number computations (i.e. symbols only). As well, although there has been a substantial body of research looking at students' strategies in solving word problems, only a small amount of this research has included research on mental computation as defined in this chapter. Carpenter, Franke, Jacobs, and Fennema (1996) investigated computational strategies in their three year longitudinal study of children's invented strategies for solving addition and subtraction word problems for 2 and 3 digit examples. It was found that students who used invented strategies demonstrated knowledge of base-ten number concepts and were successful in extending their knowledge to new situations.

Mental computation strategies

A variety of strategies for mental computation has been identified in studies of mental computation performance of children and adults. One of the earliest was a study by Ginsberg, Posner, and Russell (1981) of mental addition strategies for number computations (computations in symbolic form) presented verbally, of schooled and "unschooled" American and Dioula (Ivory Coast) children and adults. Ginsberg et al. found that there was a high incidence of error for all children who used the pen and paper algorithm mentally. They also found that the "unschooled" subjects had more success, and showed a good "number sense", using a strategy which they called *regrouping* (referred to in this chapter as *left to right separated place value)*. When using this strategy, children separate numbers using place value and calculation progresses from left to right, regrouping numbers to enable easier calculations, for example:

26 + 38:	20 + 30 = 50, 6 + 8 = (6 + 4) + 4 = 10 + 4 = 14, 50 + 14 = 64
74 - 28:	74 = 60 + 14, 60 - 20 = 40, 14 - 8 = 6, 40 + 6 = 46

Madell (1985) also identified the left to right procedure, but further reported non-columnwise calculations. He investigated Grade K to 3 children's addition and subtraction strategies for number computations in a school where computation strategies were not taught until the end of Grade 3. He documented the presence of the left to right separated strategy, but also another left to right strategy which is categorised in this chapter as *left to right aggregation*. *Left to right aggregation* is a strategy where one number is not partitioned into place values, and

the second is added in a form where it is separated into place values, the larger place values being added first. The left to right form identified by Madell is as follows:

26 + 38:	26 + 30 = 56, 56 + 8 = 64
74 - 28:	74 - 20 = 54, 54 - 8 = 46
or	28 + 40 = 68, 68 + 6 = 74, 40 + 6 = 46 (missing addend form)

There is also *right to left aggregation* where the smaller place values are added first.

Hope (1987) also identified the presence of *left to right separated place value* and *left to right aggregation* strategies for number computations in a case study of a 13 year old student who was a highly skilled mental computer. He argued that left to right strategies were memory efficient, because a series of discrete calculations does not have to be retained in short term memory and the order of the series does not have to be transformed back into a left to right sequence. He contended that it was easier to mentally combine than remember partial results, so a strategy of continually updating calculations was likely to be more efficient. Additionally, left to right strategies were identified by Kamii (1989) who studied grade 2 children who had been encouraged to invent their own computation strategies. She found that these children tended to add the tens first. She further observed that, when children added the tens first, they made less errors and demonstrated more number sense when presented with computation with misaligned digits or in a horizontal form.

Carraher et al. (1987) studied 16 Brazilian third graders' written and oral mathematics using simulated store problems, story problems and number computations (the children had been taught the pen and paper algorithm). They observed that the children tended to use mental strategies, mainly left to right, for the simulated store and word problems, and pen and paper methods for the computation exercises. Children who worked left to right produced more sensible answers. Carraher et al. also found a strategy which focused on numbers wholistically. Children using this strategy decompose the numbers in the problem (usually one number) so that they relate to tens or hundreds, use the decomposed numbers in the computation and then compensate. The decomposed numbers relating to tens and hundreds used by the children studied by Carraher et al. tended to be smaller than the numbers in the problems and compensation tended to be by adding left-overs. The following examples exemplify the techniques used by the Brazilian children:

268 + 138:	268 + 132 + 6 = 400 + 6 = 406
243 - 75:	243 - 100 + 25 = 143 + 25 = 168
	200 - 70 - 5 + 43 = 125 + 43 = 168

In this chapter, we refer to this strategy as *wholistic compensation*. It is similar to the *bridged tens or hundreds* or *used tens/hundreds* strategies identified by McIntosh (1990).

Wholistic strategies were also identified by Resnick (1986), along with the *left to right separated place value* strategy, in a case study of a child aged 7 years. However, the wholistic strategy she identified and which this chapter categorises as *wholistic levelling*, operates on both numbers together and increases and decreases numbers to take advantage of one number's closeness to hundreds, fifties or twentyfives, as the following examples show:

56 + 98: (56 - 2) + (98 + 2) = 54 + 100 = 154
172 - 47: (172 + 3) - (47 + 3) = 175 - 50 = 125

In an intervention study of Grade 7 students (Markovits & Sowder, 1994), instruction emphasised the invention of non-standard methods for mental computation. *Wholistic* strategies were also identified, and categorised as *non-standard with reformulation*. Further, the left to right process was categorised as *non-standard with no reformulation*.

Beishuizen (1993) focused on four strategies for mental addition and subtraction: *left to right separated place value*; *right to left separated place value*; *left to right aggregation*; and *right to left aggregation*. He called these strategies *1010*, *u-1010*, *N10* and *u-N10*. His definitions are summarised in Table 8.1.

Table 8.1
Description of mental strategies categorised by Beishuizen (1993)

Strategy type	Example	Method
left to right separated place value (1010)	26 + 38 74 - 28	20 + 30 = 50, 6 + 8 = 14 = 64 60 - 20 = 40, 14 - 8 = 6 = 46
right to left separated place value (u-1010)	26 + 38 74 - 28	6 + 8 = 10 + 4, 20 + 30 + 10 = 60 = 64 14 - 8 = 6, 60 - 20 = 40, 6 + 40 = 46
left to right aggregation (N10)	26 + 38 74 - 28	26 + 30 = 56, 56 + 8 = 64 74 - 20 = 54, 54 - 8 = 46
right to left aggregation (u-N10)	26 + 38 74 - 28	26 + 8 = 34, 34 + 30 = 64 74 - 8 = 66, 66 - 20 = 46

It is possible for the *left to right aggregation (N10)* and *right to left aggregation (u-N10)* strategies to have a missing addend form (not mentioned by Beishuizen), as follows:

left to right aggregation (N10) 74 - 28: 28 + 40 = 68, 68 + 6 = 74, 40 + 6 = 46
right to left aggregation (u-N10) 74 - 28: 28 + 6 = 34, 34 + 40 = 74, 6 + 40 = 46

Beishuizen (1993) conducted a field study involving 125 high and low achieving middle-class grade 2 children in Holland (where mental strategies *left to right aggregation* and *left to right separated place value* are both taught) to investigate the effectiveness of the *left to right separated place value, left to right aggregation, right to left aggregation* and *right to left separated place value* strategies for number computations. The strategies *right to left aggregation* and *right to left separated place value* were not classified separately, but included in the main categories *left to right aggregation* and *left to right separated place value* respectively. Two-digit addition and subtraction questions with and without regrouping were presented on cards. Beishuizen found that the *left to right separated place value* strategy was favoured by lower achievers but that *left to right aggregation* and *right to left aggregation* were more powerful in terms of supporting correct mental computation for larger and more complex numbers. He argued that aggregation strategies *left to right aggregation* and *right to left aggregation* seemed difficult for children to acquire spontaneously. The tendency for lower achievers to use less powerful strategies was also reported by Hope (1985), who found that poor mental computers relied almost exclusively on the pen and paper algorithm.

In a study of 32 Grade 4 children, Heirdsfield (1995) found that *right to left separated place value* was the dominant strategy for most addition and subtraction word problems and all algorithmic exercises. The next most popular strategy was *left to right separated place value*. The strategy *left to right aggregation* was seldom used, supporting the finding of Beishuizen (1993). Children used a greater variety of strategies for subtraction than addition. A possible reason for this variety for subtraction is that students' understanding of place value and regrouping procedures necessary for successful subtraction (Fuson, 1992) may not be sufficiently developed. Consequently, the children generated their own strategies for solving subtraction. Further, she found that children who used a variety of mental strategies appeared to be more proficient in computational estimation, used more advanced number facts strategies (recall and some derived facts strategies) and exhibited greater facility with numbers. She compared children who used a variety of strategies with students who employed only one strategy consistently. The children who employed only one strategy all used *right to left separated place value*, mirroring the pen and paper algorithm they had been taught. These children were less proficient in computational estimation, used more primitive number fact strategies (more count strategies), and were less flexible. This finding supports the positions of the Australian Education Council (1991) and Reys (1985), who argued that proficient mental computers are able to use a range of methods depending on the context, and the conclusions of Hope and Sherrill (1987) and Maier (1977), who found that unskilled mental computers relied almost exclusively on the traditional pen and paper algorithms.

Developmental trends

To explore the behaviour of Australian children with respect to mental addition and subtraction, 104 children of varying mathematical abilities (one-third each of above average, average and below average ability) from 6 Queensland primary schools representing a variety of social and cultural backgrounds were studied for two years (from the beginning of Grade 2 to the beginning of Grade 4). All their teachers followed the normal primary Queensland mathematics syllabus (Department of Education, Queensland, 1990, 1991). In Grades 2 and 3, this syllabus focuses on developing basic addition and subtraction facts and teaching the pen and paper algorithms for addition and subtraction. There was direct teaching of the pen and paper algorithm but no teaching of non-traditional strategies.

The pen and paper algorithm for addition proceeds as follows:

1	1		
25	25	25	25
+ 38	+ 38	+ 38	+ 38
	13	3	63

The procedure for pen and paper subtraction is similar:

1	1	1	
75	65	65	65
- 28	- 28	- 28	- 28
		7	47

These algorithms have common aspects with the *right to left separated place value* strategy for mental addition and subtraction. No other mental computation strategy has these commonalities. Therefore, a major question is whether the teaching of the pen and paper algorithms, with their attendant right to left procedure, has an impact on children's choice of mental computation strategy.

The interview tasks

The children were interviewed using Piaget's revised clinical interview technique (Ginsburg, Kossan, Schwartz, & Swanson, 1983). The interview questions consisted of a series of 2 and 3 digit addition and subtraction word problems relating to money, presented in visual and oral form (the interviewer read the questions). There were three types of problems (based on Carpenter & Moser, 1984): join addition (e.g. If the apple cost 35 cents and the orange 27 cents, how much did both cost?); separate subtraction (e.g. If John had 82 cents and spent 54 cents on bananas, how much money does he have left?); and missing addend subtraction (e.g. If Nancy had 47 cents and the chocolate costs 75 cents, how much extra money does she need?).

As far as possible, numbers for the questions were chosen to allow for a range of mental strategies to be employed. In particular, examples were chosen with numbers near 50s or 100s so that the *wholistic* strategies might be accessed. The question types used are summarised in Table 8.3. The structure of the interview was to give the questions for each problem type in increasing order of difficulty until the children's responses indicated they were experiencing difficulties. Children's responses from previous interviews assisted the interviewer find appropriate starting levels for the next interview. Children were given more difficult questions as they moved through Grades 2 and 3 to Grade 4, moving from questions without regrouping to questions which involve regrouping and moving from two digit questions to three digit questions.

Table 8.3
Addition and subtraction question types

Question types	Examples of numbers used
Join addition 2 digits, no regrouping	32 + 14
Join addition 2 digits, regrouping	42 + 49
Join addition 3 digits, no regrouping	246 + 53
Join addition 3 digits, regrouping	246 + 99
Separate subtraction 2 digits, no regrouping	48 - 25
Separate subtraction 2 digits, regrouping	95 - 47
Separate subtraction 3 digits, no regrouping	253 - 42
Separate subtraction 3 digits, regrouping	253 - 98
Missing addend subtraction 2 digits, no regrouping	86 - 30
Missing addend subtraction 2 digits, regrouping	92 - 57

The children were interviewed six times: the beginning, middle and end of grade 2 (interviews 1, 2 and 3), the middle and end of grade 3 (interviews 4 and 5), and the beginning of grade 4 (interview 6). The children were withdrawn from the classroom and interviewed in a separate room by the researcher. Children were not given pen or paper or any other calculation aids, but were allowed to count aloud or use their fingers. The interviews, which were all videotaped, were kept to approximately 30 minutes.

Children's responses

The children exhibited all the strategies found in previous research. In addition, some students always used *counting*, even when the numbers made this an impossible strategy, for instance,

55-27: "54, 53, 52, 51, ... I've lost count."

Other students used various mixtures of the strategies previously described. After intense study of the videotapes, the categories shown in Table 8.4 were found to cover all responses.

Table 8.4
Mental computation strategy categories

Strategy category	Description of student response	Examples
Counting	Students predominantly count both numbers by modelling with fingers, or count on or back with or without modelling;	**27+15**: 27, 28, 29, 30, 31, 32, ... **33-17:** 33, 32, 31, 30, 29, 28, ...
Right to left separated place value	Students predominantly take both numbers and split them using place value, then add or subtract the parts, proceeding right to left (ones before tens)	**28+35**: 5+8=13; 20+30=50; 50+13=50+10+3=63 **73-27**: 13-7=6, 60-20=40; 46
Left to right separated place value	Students predominantly take both numbers and split them using place value, then add or subtract the parts, proceeding left to right (tens before ones)	**28+35**: 20+30=50, 5+8=13; 50+13=63 **73-27**: 60-20=40, 13-7=6; 46
Aggregation	Students predominantly use left to right and right to left aggregation, where one number is kept whole and the other number is separated into place values before being added or subtracted to and from the first number.	**28+35**: 28+30=58+5=63; **28+35**: 28+5=33+30=63 OR **73-27**: 73-20=53-7=46 **73-27**: 73-7=66-20=46
Mixed	Students use a mixture of aggregation and separated place value strategies, or a mixture of separated place value strategies, involving a mixture of left to right and right to left computation, both within a question and across a question type.	**368+275**: 368+200=568; 568+5=573; 573+60=633 **423+295**: 3+5=8, 400+200=600, 628+90=718 **546-284**: 500-200=300, 6-4=2, 342-80=262
Wholistic	Students predominantly use compensation or levelling strategies, or use a mixture of wholistic and earlier strategies.	**38+56**: 40+50+4=94 **618-289**: 629-300=329 **727-286**: 700-200=500, 527-86=541-100=441

Some children used only one strategy for all questions of a certain type but a significant number did not. For some of these students there seemed no immediate reason for strategy choice; other students appeared to base their selection of strategy on the numbers given in the question. For instance, one child responded to 39 + 57 with "30 plus 50 equals 80, 9 plus 7 equals 16, so that's 96" (*left to right separated place value*) but for 36 + 29 responded "36 plus 9 is 45, 45 plus 20 makes 65" (*right to left aggregation*). The weaker students counted in most cases, and the more able students used a mixture of aggregation and wholistic strategies.

Trends

The data obtained from the six interviews are very rich and complex, and we can only summarise general trends here. More detailed results have been reported elsewhere (Cooper, Heirdsfeld, & Irons, 1995a, 1995b).

In interview 1 (the first interview in Grade 2), only 2 digit examples were presented to the children and only examples without regrouping were attempted by reasonable numbers of children. The most used successful strategy was *counting*. However, this strategy was never used for the more complex regrouping exercises, where *left to right separated place value*, *aggregation*, and *wholistic* were successfully employed. By interview 2, the percentage using *counting* decreased and the percentage using *left to right separated place value*, *right to left separated place value*, and *aggregation* increased, *left to right* consistently resulting in higher performance than *right to left*. This trend continued in interview 3 (the last interview of Grade 2).

In interview 4 (the first interview in Grade 3), the first attempts at 3 digit examples appeared. These more difficult 3 digit examples, although attempted successfully by only a few students, were completed more adeptly by employing *wholistic* strategies. Except for 2 digit addition with regrouping, *left to right separated place value* resulted in consistently higher performance than *right to left separated place value*. There was a significant minority successfully using *aggregation*, *mixed*, and *wholistic* strategies. In interview 5 (at the end of Grade 3), 2 digit no regrouping exercises were most successfully completed using *left to right separated place value*, whereas 2 digit with regrouping exercises were most successfully completed with *right to left separated place value*. The most complex examples, 3 digit addition and subtraction and 2 digit regrouping missing addend, were most successfully completed by employing *wholistic*, *aggregation*, or *mixed strategies*.

At the beginning of Grade 4, in interview 6, *right to left separated place value* became the most successfully employed strategy category, except for subtraction without regrouping (both separate and missing addend) where *left to right separated place value* resulted in higher performance.

Addition question types, particularly two digit without regrouping, were the simplest ones presented to the students and the easiest for them to complete successfully. The number of students successfully attempting addition questions increased from interviews 1 to 6, even though the difficulty of the questions increased. At the same time, successful use of the *counting* strategy decreased and successful use of the *left to right separated place value*, *right to left separated place value*, *aggregation*, *mixed*, and *wholistic* strategies increased, peaking at interview 4. *Left to right separated place value* became the most used successful strategy by interview 3, but then declined and was replaced by *right to left separated place value* by interview 6.

The harder addition question types (3 digit) were initially only attempted by a small group of able students who used high level strategies (e.g. *wholistic*). As the children progressed through Grades 2 and 3, more children attempted these questions and a wider range of strategies was used. By interview 6, the *right to left separated place value* strategy was used by most students to successfully solve the more complex addition questions. However, it is interesting to note that in interview 4, no student used *right to left separated place value* successfully to solve a 3 digit addition question.

Separate subtraction questions had a similar use distribution as addition questions, although fewer students were able to successfully complete the questions at each interview compared with those for addition. The percentage successfully attempting these questions increased throughout the interviews. *Counting* was the most successfully employed strategy in interview 1, but was quickly replaced by *left to right separated place value* and *aggregation*. However, by interviews 5 and 6, *right to left separated place value* resulted in highest performance. Like addition, *counting* was rarely successfully employed at this point.

Like addition, *wholistic* was the most successful strategy for the more difficult separation subtraction questions in interviews 4 and 5 (along with *aggregation* in interview 5), and *right to left separated place value* resulted in highest performance in interview 6. Further, like addition, *right to left separated place value* was not used successfully by any students in interview 4.

The results for missing addend subtraction questions showed similar trends to the separate subtraction questions. In interview 1, *counting* was the most successfully used strategy, but was soon replaced by *left to right separated place value*, which resulted in highest performance for the remainder of the interviews. However, use of the *right to left separated place value* strategy increased by interview 6 and resulted in highest performance for more complicated examples. Further, like separation, the percentage correctly attempting the question type increased until interview 6 when more students were presented with the regrouping examples. Compared with separation subtraction, fewer students were able to successfully complete missing addend questions at each interview.

Discussion

The interviews showed that children used a variety of strategies and that these strategies changed across the interviews. Moreover, the percentage of children successfully attempting more difficult questions increased over the interviews. The percentage of children correctly using the *counting* strategy decreased over the interviews (although it was still evident in interview 6 for the simpler examples attempted by the poorer students), while the percentage of children correctly using higher level strategies increased. *Aggregation* strategies were less

popular than *separated* strategies. The preference for *separated* strategies over *aggregation* was also reported in the study by Beishuizen (1993).

The most interesting change was in the successful use of *right to left separated place value*. *Left to right separated place value* or *aggregation* became the most successful strategies for most question types by interview 2, but by interview 6 *right to left separated place value* was the most successfully employed (except for the simpler question types). It is interesting to speculate on the reasons for *right to left separated place value* replacing *left to right separated place value* and *aggregation* as the most used successful strategy, because *left to right separated place value* and *aggregation* would appear to be more efficient and effective strategies. *Right to left separated place value* is similar to the pen and paper algorithm procedure for 2 and 3 digit addition and subtraction which is a major part of instruction in Queensland in Grades 2 and 3. The growth in the use of *right to left separated place value* may be a result of instruction. This interpretation is supported by Heirdsfield (1995), who revealed that *right to left separated place value* is the only strategy which students use solely.

With regard to question types, correct attempts of missing addend subtraction questions were significantly fewer than addition and separation subtraction questions, indicating the difficulty that children have with this meaning of subtraction. Further, children were less successful with subtraction than addition, suggesting difficulties with the concept of subtraction in general. Strategy varied according to the presence or absence of regrouping and the question type. In interview 6, *left to right separated place value* resulted in the highest performance for separation and missing addend subtraction without regrouping; while *right to left separated place value* was most successfully used for 2 and 3 digit addition and subtraction with regrouping.

The strategy results seen in the interviews give rise to interesting inferences. The students tended to progress to the higher level strategies and the students attempting harder examples earlier tended to use higher level strategies and were more accurate. It could be argued that the higher level strategies are more powerful and efficient in terms of their use of children's cognitive capacity and this could be the reason for the students with greater capability using them. However, the results are not unequivocal in this. Greater accuracy with higher level strategies could be due to reduced cognitive load or more capable children using them. As well, there is the increased use of the *right to left separated place value* strategy, particularly for regrouping example types. This is a reduction in strategy power. It could be teaching related, but is it? Evidence from this study appears to support this, as the dominance of this strategy became particularly noticeable after the direct instruction of the pen and paper algorithm. The teasing out of these issues will require a deeper analysis of the data and individual student preferences to be followed across example types and interviews.

However, some things were unequivocal. Many children could efficiently and effectively compute with their own non-traditional mental algorithms before instruction in pen and paper algorithm procedures and many children preferred and used a variety of approaches.

Children chose their strategies depending on the difficulty of the numbers and the word problem type. These findings need to be taken into account in planning teaching for the operations of addition and subtraction.

Teaching implications

The research above has shown that mental computation involves a variety of methods. This is similar to the findings of Carraher, Carraher, & Schliemann (1985). Clarke and Kelly (1989) have argued that 80% of mathematics used in everyday life is done mentally. This suggests primary schools should teach mental computation in a way that promotes flexibility of methods. The research above has also shown that children who are proficient in mental computation use a range of methods flexibly in different contexts and that low proficiency students almost exclusively rely on *counting* and *right to left separated place value*, the strategy similar to the pen and paper procedure. Reys (1985) argued that proficient mental computers use a range of methods flexibly, while Hope (1985) concurred that low proficiency computers use counting and pen and paper procedures. Heirdsfield (1995) reported that students who use a variety of mental strategies creatively use number properties and are more proficient in estimation and number fact strategies. This suggests that low performing mental computation students may benefit from direct teaching of mental strategies other than *right to left separated place value*.

In particular, the study shows that, other than the pen and paper related *right to left separated place value*, the most correctly used addition and subtraction mental strategies were *left to right separated place value* and *aggregation*. This is interesting as children were not instructed in these "alternative" strategies; yet they were able to successfully employ these strategies. Further, *aggregation* strategies were little used. These results were supported by Heirdsfield (1995) and Beishuizen (1993). This suggests that it might be useful to explicitly explore *aggregation* strategies. Overall, the research findings suggest that either explicit teaching of pen and paper procedures must be reduced or explicit exploration of mental algorithms should begin, particularly the more efficient and powerful *aggregation* and *wholistic* strategies.

Several researchers have discussed possible alternative teaching techniques. Kamii, et al. (1991) advocated constructing number concepts "from the inside". Lindquist (1989) reported that what students learn and what teachers had intended they should learn are often in conflict. This means that care should be taken to ensure that student learning and teacher teaching are not in conflict. The research findings above indicate that children can develop their own efficient and proficient strategies spontaneously, without instruction. Similar findings were reported by from Carpenter et al. (1996). As a consequence of this, classroom activities to facilitate efficient mental computation should prevent naturally developing "spontaneous" strategies from being repressed and should directly develop effective mental strategies

(Heirdsfield & Cooper, 1995). Further, support for not teaching algorithms at all comes from Carpenter et al. (1996), who suggest that invented strategies could form a basis for developing understanding of processes and concepts, and calculators be used for more complex calculations.

Aggregation and *wholistic* strategies do not have both numbers separated by place values. Hence the normal structured material (for instance, Multibase Arithmetic Blocks) are not applicable. However, two teaching techniques appear capable of lending some assistance. The first of these is the number line of Beishuizen (1993) where, to find 46 + 39, children "hop along" the line from 46 to 56 to 66 to 76 (adding 30) and finally another 9 to 85, and to find 63 - 28, "hop back" 63 to 53 to 43 (subtracting 20) and finally another 8 to 35. The line also allows the children to "hop forward" from 28 to 63 with hops of 2 (to 30), 30 (to 60) and 3 (to 63), or to "hop back" 30 (to 33) and then "hop forward" 2 (to 35) to find 63-28. The second of these is similar to the number line on a hundreds board (Beishuizen, 1993; Cobb, 1995; Thornton, Jones, & Neal, 1995).

It appears imperative that children be given a chance to build on their own natural skills by choosing and verbalising mental strategies, and by choosing when mental computation is appropriate. Less emphasis should be placed on traditional pen and paper algorithms and more emphasis on identifying and developing children's legitimate spontaneous strategies.

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Chapter 9

Children's Representations Of Multiplication And Division Word Problems

Joanne Mulligan and Michael Mitchelmore

In the past decade there has been a considerable amount of research indicating that even young children are able to solve mathematical problems informally, by representing them in a number of different forms. There is an extensive body of research showing that young children can solve a variety of semantic types of addition and subtraction word problems by directly modelling the action or relationship described in the problems (Carpenter & Moser, 1984; Fuson, 1992). Recent studies on children's solutions to multiplication and division problems have also indicated that children as young as kindergarten age can solve a variety of problems by combining direct modelling with counting and grouping skills, and with strategies based on addition and subtraction (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Kouba, 1989; Mulligan, 1992).

Other recent studies analysing the growth of number concepts and processes, indicate that children's representations of problem-solving situations are closely linked to their conceptual understanding and the way they construct mathematical relationships (Davis, 1992; Davis, Maher, & Noddings, 1990; Goldin & Herscovics, 1991; Hiebert & Wearne, 1992; Maher, Davis & Alston 1993; Thomas, this volume; Thompson, 1992). Researchers have also highlighted children's representations as evidence of the idiosyncratic and creative ways in which they structure mathematical relationships (Pirie & Keiren, 1993; Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, 1994).

Some researchers suggest that conceptual understanding is built on the notion of constructing connections between representations of mathematical ideas (Hiebert & Wearne, 1992; Hiebert & Carpenter,1992; Lampert, 1990, 1992; Lesh, Post & Behr, 1987). This involves building relationships between quantities that are represented physically, pictorially,

verbally and symbolically (Hiebert & Wearne, 1992). Moreover, they suggest that building connections between external representations supports the growth of internal conceptual understanding and more coherent internal representations of mathematical ideas (Hiebert & Carpenter, 1992).

However it is not entirely clear how children use their external or internal representations in building conceptual understanding. If children's representations of their computational strategies are a reflection of their developing concepts, or if their representations actually enhance the development of mathematical concepts and processes, then their representations may provide valuable evidence of how conceptual understanding grows. This chapter reports an exploratory study examining how third graders spontaneously represented and calculated solutions to multiplicative word problems as they worked in a mini-classroom situation in which they were required to represent and explain their solution strategies.

Background

Studies analysing the actual strategies young children correctly use to solve whole number multiplicative problems have yielded fairly consistent results (Anghileri, 1989; Bryant, Morgado & Nunes, 1993; Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993; Clark & Kamii, 1996; De Corte, Verschaffel & Van Coillie, 1988; Kouba, 1989; Mulligan, 1992; Steffe, 1992). Broadly, strategies have been classified in two ways: calculation strategies, and modelling strategies. Calculation strategies involve increasingly sophisticated counting: direct or unitary counting, rhythmic and skip counting, additive strategies based on repeated addition, and multiplicative strategies. (For descriptions, see Mulligan & Mitchelmore, in press). Modelling strategies primarily involve the "use of physical objects" (Kouba, 1989). Carpenter et al. (1993) reported the use of direct modelling by Kindergarten children as they learned to represent and solve simple multiplication and division word problems. Modelling involves using objects such as tokens or fingers, or drawing ikons or tallies; some problems are apparently modelled internally. All these modelling strategies have been reported as occurring in conjunction with all the calculation strategies.

In a 2-year longitudinal study of 60 children in Grades 2 and 3, Mulligan and Mitchelmore (in press) identified children's solution strategies to a range of multiplicative problems and inferred their intuitive models of multiplication and division. (Children were allowed to use cubes to model the problems but no written or drawn responses.) In that study, intuitive models were described as internal mental structures that corresponded to different types of calculation strategies. The results showed a consistent progression in the intuitive models used by the children from direct counting, to repeated addition or subtraction, to multiplicative operations. Children's conceptual understanding of multiplication and division, shown through their mental strategies, developed largely as a result of their recognising the equal group structure common to all multiplicative problems. The model employed to solve a

particular problem did not reflect any general problem feature, but rather the mathematical structure which the student was able to impose upon it. The semantic structure of problems was found to influence performance with some problem types being more difficult than others, but the semantic structure of the problem did not necessarily correspond with the strategy used to solve it.

These findings raised further questions about the way young children might be able to solve multiplicative problems if they were encouraged to represent their solution strategies more explicitly through concrete modelling, pictorial, ikonic and notational recordings, as well as verbal and written explanations. It was also questioned how children might represent multiplication and division problems in a classroom setting where collaboration between pupils and the teacher was encouraged, rather than a researcher examining children's solution strategies in a clinical setting.

Classroom studies

There has been a number of studies on teaching and learning multiplication and division focusing on the ways in which children represent related problem situations (Boero, Ferrari & Ferrero; Burns, 1989; Carpenter et al. 1993; Lampert, 1990; Murray, Olivier & Human, 1992; Nesher, 1988, 1992; Rathmell & Huinker, 1989).

Lampert (1990) used an instructional approach based on problem solving intended to help children make sense of the process of multiplication and division with numbers larger than 10. This included telling and illustrating number stories, working on problems whose solutions require various ways of grouping and counting, and solving problems drawn from "real world" situations. An important feature of Lampert's style is how she encourages children to connect their representations to the symbols used in formal school mathematics. Burns (1989) presented multiplication in such a way that children were involved in thinking about grouping, investigating patterns, writing multiplication stories, relating multiplication to geometry, and patterning on a multiplication chart. Another approach, focusing on part-whole relationships was implemented by Rathmell and Huinker (1989) who investigated children's representations of multiplication and division problems through the use of materials and diagrams. Another problem-centred learning project (Murray, Olivier, & Human, 1992), also showed how children can represent multiplication and division problems meaningfully and develop autonomy and adaptability in the learning situation.

However, more in-depth research is needed to identify the internal mental structures that children use in developing conceptual understanding of multiplication and division, and to elucidate the growth of effective strategies for solving a range of multiplicative problems. Examining children's drawn and written representations of multiplicative situations is one way of gaining insight into this complex process. Some of the difficulties children face in developing multiplicative reasoning might be more clearly identified, and appropriate

instructional strategies developed. The present study was therefore designed to answer the following questions:

(i) Can previously identified intuitive models of multiplication and division (direct counting, repeated addition or repeated subtraction, and multiplicative operations) be identified in children's representations of multiplicative problems?

(ii) Does the semantic structure of the problem influence children's representations, including their symbolisations, or do children develop and apply a set of strategies which they can apply to all problems?

Method

The methodology was based on constructivist principles of learning exemplified by the Cognitively Guided Instruction Project (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Cobb, Wood, Yackel, & Perlwitz, 1992). The researcher assumed the teaching role of a facilitator, systematically providing structured mathematical problems for children to represent and solve in a mini-classroom setting. The study also has some similarities to the constructivist teaching experiment (Steffe, 1991) in that it allowed the researcher to act as teacher in the investigation of children's mathematical thinking.

Design

Ten 2 hour teaching sessions were conducted with ten Grade 3 female students, once a week for ten consecutive weeks of the third school term. It was considered that ten subjects would be manageable as a small group in the research setting, but could still provide a model for regular classroom practice. The classroom teacher reported that by the beginning of the teaching sessions the students had been instructed in basic addition and subtraction, and in basic multiplication facts involving the 2- to 10- times tables, but not related division facts. The students were not instructed in solving multiplication and division word problems in the regular classroom prior to or during the study.

Sample

The sample consisted of ten girls ranging in age from 8 to 9 years randomly selected from 21 girls in the third grade of a Catholic school in the Sydney Metropolitan area. There was a wide range of mathematical ability represented across the sample and students were from varying socioeconomic and ethnic backgrounds. The sample was limited by gender in order to eliminate possible gender differences.

The problems

The problems were based on those used in an earlier longitudinal study referred to earlier (Mulligan, 1992, Mulligan & Mitchelmore, in press). The multiplication problems represented four of the ten semantic structures identified by Greer (1992), Nesher (1988), Vergnaud (1988), and others: equivalent groups, multiplicative comparison, cartesian product, and rectangular array. Division problems were two inverses of the equivalent groups structure and an inverse rate structure. A further partitive division problem, called sub-grouping, was included to explore how children would approach a situation where the divisor was larger than the dividend. All problems were set in contexts which were familiar to young children.

Two categories of number size were selected in order to compare the influence of small and large combinations on performance and strategy: The product either lay between 4 and 20 ("small numbers") or between 20 and 40 ("large numbers".) These number domains were used because the Grade 1 curriculum focuses on whole number combinations between 0 and 20 and extends to 100 by Grade 3. Two examples of each problem structure were employed in the teaching project as shown in Table 1.

Table 9.1
Multiplication and division word problems

Multiplication	Division
Equal Groups There are 2 tables in the room with 3 children seated at each table. How many children are there altogether (in the room)? There are 6 plates of chocolate crackles with 4 crackles on each plate. How many chocolate crackles are there altogether?	**Partition** There are 12 children and 6 tables in the room. How many children are seated at each table? There are 24 cakes and 3 plates to put the cakes on. How many cakes can go on each plate (if you put the same number on each plate)?
Comparison There are 3 books on Sue's table and 5 times as many on the other table. How many books are on the other table? Samantha has 3 pencils and Rebecca has 9 times as many. How many pencils does Rebecca have?	**Rate** If it costs 18c for 3 ice blocks, how much does it cost for 1? If it costs 35c for 5 biscuits, how much does it cost for 1?
Cartesian Product There are plain chips or chicken chips in small medium or large packets. How many different packets of chips can you choose from? There are 3 sizes of ice cream cones (small, medium, large), and 4 flavours (e.g. vanilla, chocolate, strawberry, banana). How many different choices of icecream cones can you make?	**Quotition** There are 18 children and 3 children are seated at each table. How many tables are there? There are 36 toys and each child got 3 toys. How many children are there?
Array There are 4 lines of children with 3 children in each line. How many children are there altogether? There are 7 lines of children with 3 children in each line. How many children are there altogether?	**Sub-grouping** There are 6 apples to be shared evenly between 12 people. How much apple will each person get? There are 12 apples to be shared evenly between 24 people. How much apple with each person get?

Procedures

The researcher conducted all the teaching sessions in a separate classroom used for specialist teaching. The children were matched in pairs, on the basis of ability determined by the classroom teacher. The children were seated at separate tables in their specified pairs consistently for the ten sessions. It was anticipated that pairs of children would work co-operatively, but the children were free to work independently if they desired.

Problem-solving sessions

The teaching sessions involved the children in solving and representing the multiplication or division word problems shown in Table 9.1. Problems were written on cards and presented to the children at each table, and read aloud to the children as a group, or individually, by the researcher. A range of concrete materials and pencil/paper were made available. The researcher explained to the children that they could use the materials to assist in solving the problem if they wished. Children were required to record their solution process with pencil/paper.

Over the ten week period, each problem was presented in both small and large number versions. The multiplication small number problems were presented first followed by the division small number problems, in the order shown in Table 9.1. The problems with large number combinations were asked after the small number problems. Children were encouraged to work at their own pace so that some children completed more problems than others in each session. Each child completed at least two examples of each problem type over the period of the teaching sessions.

Children were asked to solve and represent the problems in the mode or modes of their choice, but the researcher also encouraged the following methods of response:

(i) modelling the problem with counters or fingers;

(ii) drawing a picture or diagram to represent the solution process;

(iii) representing the problem or solution using notation or as a number sentence, and

(iv) explaining how they solved the problem verbally and in writing.

The researcher did not teach any specific problem-solving strategies. It was judged important that the problem-solving process be as natural as possible, so the researcher's main role was to encourage a variety of modes of representation. During the sessions each child was asked, in turn, to explain to the researcher (at a separate table set up for interviews) how they solved the problems that they had solved in that session. The researcher recorded these responses on a record form, and collected pictorial and written samples of solutions. (For details of questioning techniques see Mulligan & Mitchelmore, in press). The children

discussed their problem-solving strategies with their partner, with other children and with the researcher. Children were encouraged to develop co-operative learning strategies where they learned to help another child without giving the answer. The researcher questioned the children about the similarities or differences between their representations of the problems. For example, did the child use concrete materials to model each problem, or did they draw pictures of the equal grouping structure of the problem in the same way? Children using the formal notation of multiplication and division were asked if they had recorded all of their problems in this way.

Analysis of Representations

There has been a number of approaches used by researchers to analyse children's representations of mathematics in problem-solving situations. Various modes of representation were described by Lesh et al. (1987) in terms of children's interpretation of real life situations: spoken or written language, concrete objects, pictures, or mathematical notation. These aspects are common to many studies analysing children's representations (Outhred, this volume; Maher et al., 1993; Rubin & Russell, 1992; Thomas & Mulligan, 1995; Thompson, 1992).

Goldin's model for children's problem-solving competencies distinguishes cognitive representational systems internal to problem solvers (a theoretical construct to describe the child's inner cognitive processing) from (external) task variables and task structures (cf. Goldin & McClintock, 1984). Three of the five types of internal representational systems discussed by Goldin contain features that can be particularly useful in analysing children's representations of word problems: (a) verbal/syntactic systems (using mathematical vocabulary, developing precision of language, self-reflective descriptions); (b) imagistic systems (non-verbal, non-notational representations, e.g. visual or kinaesthetic); and (c) formal notational systems (using notation, relating notation to conceptual understanding, creating new notations).

Similarly, children's external pictorial, ikonic and notational external representations were analysed in conjunction with their verbal and written explanations, and the researcher's observation notes and anecdotal comments. The external representations were examined according to three dimensions: the type of representation (pictorial, ikonic, notational), evidence of structural development of multiplication and division concepts, and identification of intuitive models for multiplication and division (Mulligan & Mitchelmore, in press). The analysis of the representation was then compared with the child's verbal and written explanation of the solution strategy.

Type of external representations

Pictorial recordings were defined as drawn or verbal depictions of objects used by the child to model the problem concretely (for example, a drawing of children seated at tables, or a description of children standing in lines.) Ikonic recordings were defined as more abstract

recordings of objects drawn as tally marks, dots, dashes, lines or geometric shapes, where the marks stood for real objects (for example, tally marks circled to show equivalent groups of people). Notational recordings were identified by the use of numerals and mathematical symbols (for example, number sentences 3+3+3+3 = 12 or 3 x 4 =12).

Intuitive models of multiplication and division

The children's internal conceptual structures of multiplication and division, described earlier in terms of intuitive models (Mulligan & Mitchelmore, in press), were inferred from elements such as grouping, partitioning, counting and patterning in their external representations, and their explanations of these representations. Representations of multiplication showed evidence of three intuitive models: (i) modelling the problem by *direct counting* one by one, without taking into account the structure of the equal sized groups; (ii) *repeated addition* of equivalent groups by rhythmic, skip counting or adding, and (iii) the *operation of multiplication*, represented as a binary operation. Intuitive models of division resembled those found for multiplication: (i) a rudimentary level, *direct counting*, where children shared and counted one by one; (ii) *repeated subtraction*, where the child started with the dividend and successively took away the numbers in each group, (iii) *repeated addition*, where the child starts at 0 and builds up to the dividend by counting or repeated adding, and (iv) the operation of multiplication as the inverse of division.

The data presented in this chapter feature selected examples of the children's representations of each problem type and their method of solution. The way individuals linked ikonic, pictorial and notational attributes within a representation, and displayed calculation strategies reflecting intuitive models of multiplication and division will be highlighted.

Results and Discussion

In most cases, children were able to solve each small and large number multiplication and division problem without difficulty, with minimal guidance from the researcher. There were wide variations in the pace at which individuals solved the problems and the strategies they used. Rhythmic and skip counting strategies were found consistently across problem types and proved to be effective for those children who could relate multiple counting patterns to their representation and solution strategy. There was a wide range of representations used to solve the problems; the most frequently used intuitive model was repeated addition. In this paper we only report a variety of representations made by the children.

Representations of Multiplication Problems

Equal groups

Figures 9.1-5 show five children's external representations of equivalent grouping problems. Figure 9.1 shows Michelle's drawing of "two tables with three children at each table." She formed equivalent groups but focused on her image of the situation as "4 girls and 2 boys," using unitary counting and addition to calculate an answer. This was meaningful for Michelle, but her internal image and notation did not reflect the structure of multiplication. For the same problem, Rebecca (Figure 9.2) calculated 3 + 3 = 6 and used an inventive pictorial and notational representation of equivalent groups. In this example, Rebecca uses equivalent groups and an intuitive model of repeated addition, even though she uses the multiplication sign. The representation also revealed her visualisation of an algorithm in vertical form. Rebecca explained "that this was the way she remembered doing sums." This examplifies how Rebecca connected previously organised mathematical representations of addition with newly constructed meaning for multiplication. She was able to generalise this vertical form of notation in her representations of all multiplication problems.

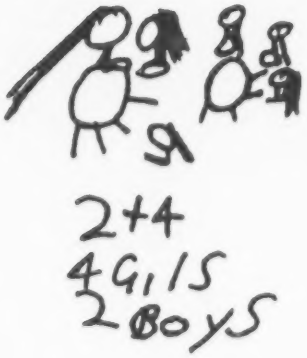

Figure 9.1: Equivalent groups (Michelle)

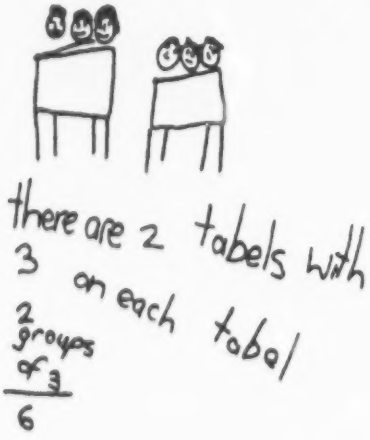

Figure 9.2: Equivalent groups (Rebecca)

In Figure 9.3 Natalie represented the equivalent grouping problem ikonically, in a linear formation, showing clearly the multiplicative structure of repeated equivalent groups. It will be noted that a linear formation was often used as shown later in Figure 9.12. Natalie's notation, reflected the use of repeated addition (where she added on four successively). Figure 9.4 shows that Becky also used this structure of multiplication but applied a skip counting pattern of fours to gain a solution more efficiently.

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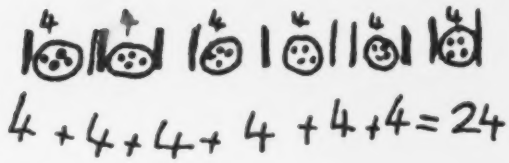

Figure 9.3: Equivalent groups (Natalie)

Figure 9.4: Equivalent groups (Becky)

In comparison, Figure 9.5 shows a typical pictorial example of an equivalent groups structure, accompanied by multiplication notation. Where children drew equivalent groups as well as using the multiplication symbol, the representation was usually expressed as "groups of" rather than "multiply". Similarly, Rebecca used this language in Figure 9.2 to describe the operation of multiplication.

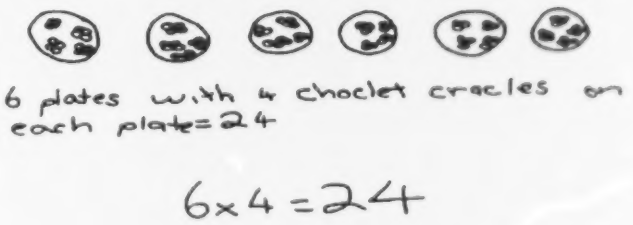

Figure 9.5: Equivalent groups (Catherine)

Comparison

The comparison problems revealed how children interpreted the semantics of "times as many". Figure 9.6 shows how Natalie used equivalent grouping and rhythmic counting to structure the problem. However, direct counting seemed to dominate her thinking despite her using formal notation of 5 x 3 =15. In this example, Natalie appears to be using all three intuitive models, direct counting, repeated addition and multiplication as an operation. Although Natalie used notation of 3 x 5 =15, she relied on forming and drawing equal groups, and direct counting to obtain an answer. In this case the use of multiplication notation is only a summarised form of repeated addition and does not mean that Natalie was using multiplication as a binary operation.

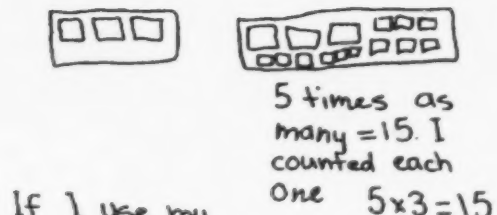

Figure 9.6: Comparison (Natalie)

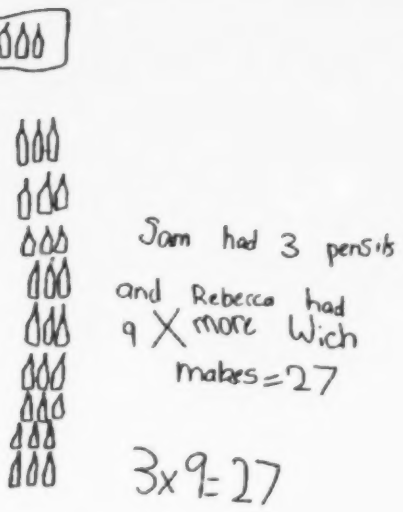

Figure 9.7: Comparison (Samantha)

Samantha demonstrated multiplication as an operation in Figure 9.7, by multiplying one group of three, nine times and it was observed that she did this without using direct counting or repeated addition to obtain an answer. Samantha, like Natalie, formed one group initially, "to be timesed." This may be significant in that many, children who were developing multiplication as an operation eventually formed one group to represent the multiplier.

Array

The array small and large number problems were modelled and drawn in pictorial, rectangular formations and solved using direct, rhythmic and skip counting, or repeated addition. Figure 9.8 shows how Catherine represented the problem ikonically by the array of 7 by 3; despite using notation 7 x 3 = 21, she used direct counting in threes rather than the operation of multiplication.

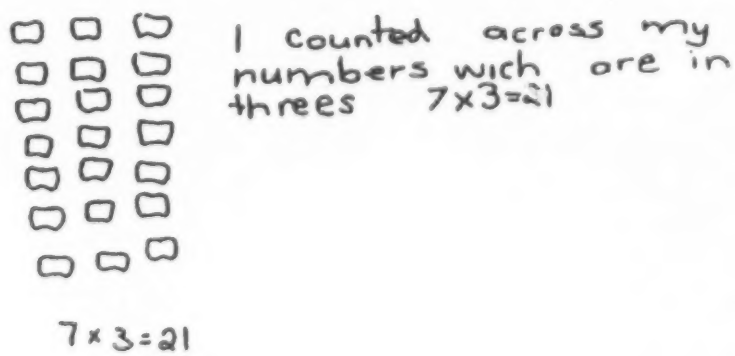

Figure 9.8: Array (Catherine)

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Natalie used the same layout but represented the "leaders" of each line distinctively. She also used a rhythmic counting pattern and repeated addition (see Figure 9.9).

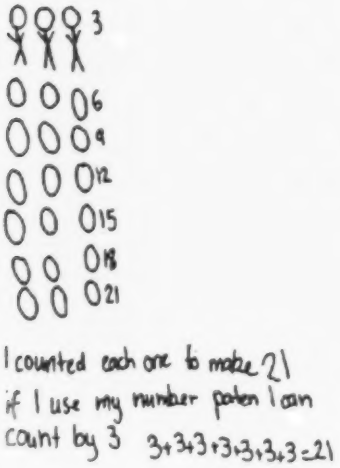

Figure 9.9: Array (Natalie)

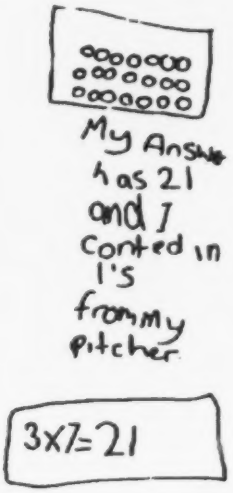

Figure 9.10: Array (Samantha)

In comparison, other children represented the array structure by drawing 3 x 7 = 21 as 3 (vertical) and 7 (horizontal). Figure 9.10 shows how Samantha did this. Note that she uses the notation of multiplication but calculates the solution using direct counting.

The representations for the array problem revealed that the use of an array formed in rows and columns, was an effective method for structuring and calculating multiplication. Some children could not make efficient use of the array structure because they formed rows and columns in disarray. However, even those children who drew arrays effectively did not connect this formation to an operation of multiplication. Outhred (this volume), also cites examples of this.

Cartesian product

Eight of the ten children were able to represent all of the cartesian product situations correctly by visualising the situation and drawing their pictorial image of the combinations. Figures 9.11 and 9.12 compare the representations made by Samantha and Catherine. Both examples show how the children drew the ice creams in order of size and categorised by flavour. Samantha's example was recorded in a vertical array formation symbolised by 4 x 3 =12, whereas Catherine drew the combinations in a linear formation. Samantha explained that her recording "was just like the one she did for the array." Catherine had previously used multiplication symbolism in a comparison problem as 4 x 3 = 12, but in this situation, chose a multiple pattern of threes to calculate the answer by repeated addition. She was unable to recognise that the problem required the operation of multiplication. Catherine focused on the way the ice creams would look like if they were served "along the counter." She explained that "it was easier to check each one that way."

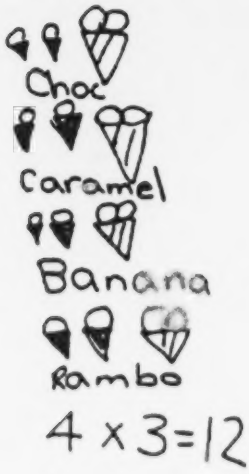

Figure 9.11: Cartesian product (Samantha)

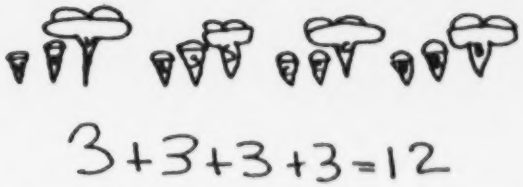

Figure 9.12: Cartesian product (Catherine)

Interestingly, most Grade 3 children attempting the problem in the longitudinal study (Mulligan & Mitchelmore, in press) simply added the two factors 4+3 = 7. Interestingly, all ten children in the present study were able to use physical, pictorial and ikonic representations, to solve the cartesian problems correctly. In most cases, children discussed their solutions with their peers and with the teacher and this assisted the child to find and record a solution. It was apparent to the researcher that the problems were solved when the child explained their thinking strategies and represented them, except where children immediately recalled a known fact and then represented it.

In summary, the representations of the equivalent grouping, comparison, array and cartesian product problems revealed a variety of grouping, counting, additive and multiplicative strategies. Almost all children showed an understanding of the multiplicative structure of all problem types by representing equal groups. However, calculation strategies and notation varied considerably. Two distinct meanings for the multiplication sign were shown (e.g. 3 x 2 as "3 groups of 2" or "3 multiplied by 2") where individual children were not necessarily consistent in their meaning and representation of the multiplication sign. The development of multiplication as a binary operation was revealed most closely by the comparison problem.

Representations of division problems

There was a wide variety of pictorial, ikonic and notational recordings made and this was consistent with the range of representations shown for multiplication problems. Similarly, the most frequently used intuitive model was repeated addition, again shown clearly by use of additive procedures.

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Partition

The structure of partitive division is exemplified by the representations made by Samantha and Michelle in Figures 9.13 and 9.14 respectively. Samantha used the more abstract representation of division notation. She began by forming the dividend, which she then partitioned into 6 equal groups; this was characterised as a repeated subtraction model of division, because she explained a taking-away process in groups rather counting back directly.

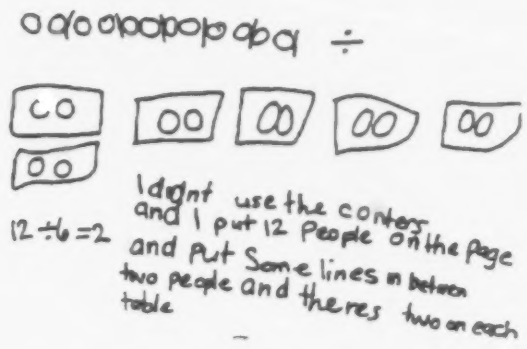
12 ÷ 6 = 2
I dont use the conters and I put 12 People on the Page and Put Some lines in between two people and theres two on each table

Figure 9.13: Partition (Samantha)

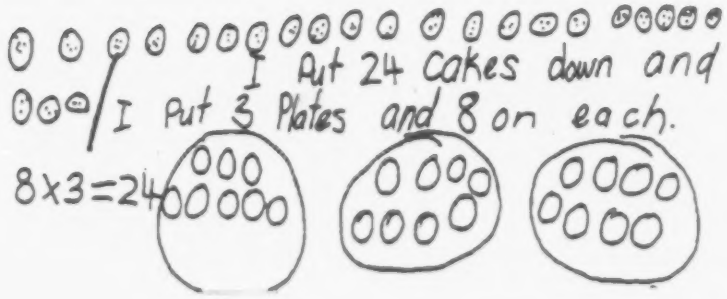

Figure 9.14: Partition (Michelle)

Figure 9.14 shows how Michelle used repeated subtraction by counting out 24 first, before forming equal groups by estimating the number required for each group and then counting to check. She recorded the result using multiplication as the inverse of division.

Rate

The rate problems were solved mainly by direct counting and repeated addition. For both small and large number combinations, children correctly estimated and grouped predominantly with counters by one-to-many correspondence. Figure 9.15 shows that Katie used one-to-many correspondence and repeated addition based on rhythmic counting in sevens. Others correctly used division or multiplication symbolism (35/7, 18 /3 or 7 x 5, 3 x 6) in an operational model. It was common for children to draw their model of the problem usually in an equivalent groups structure and then indicate the method of calculation and recording of solution. Children who used multiplication and division as an operation usually recorded

formal symbolism, but in some cases children used the operation model but recorded the solution as repeated addition or subtraction.

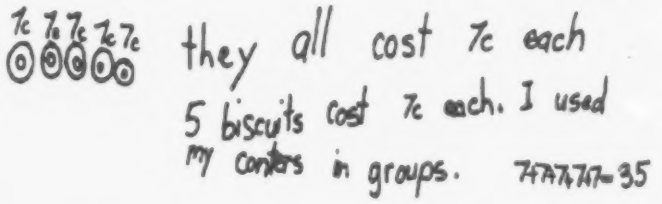

Figure 9.15: Rate (Katie)

Quotition

The children's representations of quotition problems showed evidence of direct counting, repeated addition and repeated subtraction models of division with repeated addition used most frequently. Several children counted out the dividend first and successively forming equivalent groups until there were no children left. Figure 9.16 shows how Natalie represented this and symbolised the problem as 6 x 3 = 18, "6 tables with 3 in each." Again, Natalie uses multiplication notation to represent a multiplicative situation but she relied on direct counting and drawing of each "child" to calculate the answer. Thus, she is not really using multiplication as an operation.

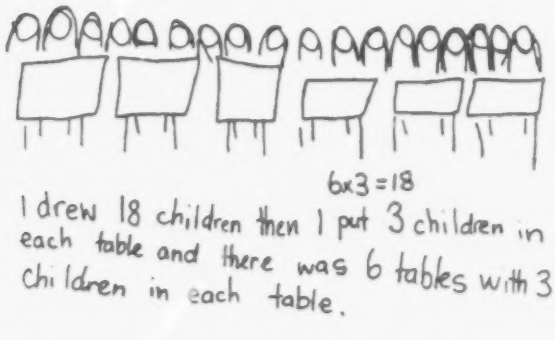

Figure 9.16: Quotition (Natalie)

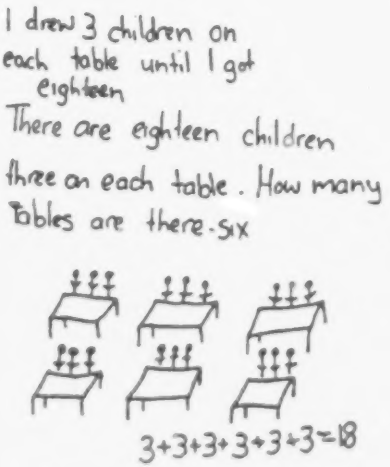
I drew 3 children on each table until I got eighteen
There are eighteen children three on each table. How many Tables are there. six
3+3+3+3+3+3=18

Figure 9.17: Quotition (Katie)

In contrast to direct counting and repeated subtraction, six children used repeated addition to solve the problem. Rather than counting out 18 first they built-up in threes to 18, as shown in Figure 9.17. Katie grouped correctly and represented the tables in one-to-many correspondence. However, she did not need to count up the number of tables because she used double counting in reference to her pictorial representation.

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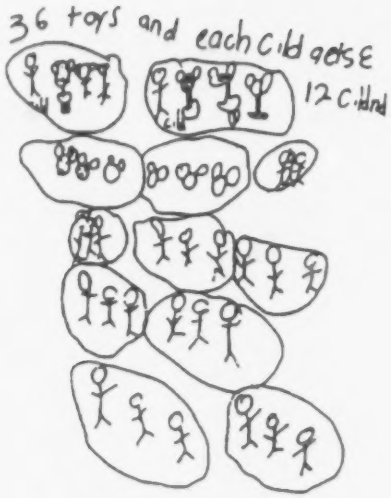

Figure 9.18: Quotition (Michelle)

Michelle generally used more immature and idiosyncratic representations than all the other children in the study. Figure 9.18 shows her representation of a quotition large number problem where 36 toys are grouped in threes. Although she began by matching "one child to 3 toys" and continued by correctly drawing three toys in each group, she changed her representation of toys to "children" for the remaining groups, interchanging the role of the multiplier and multiplicand. However, she correctly stated that there were 12 children and "each child gets three."

Generally, children symbolised the quotition problems with repeated addition or known multiplication facts. There were few attempts made to symbolise using the division sign. Children who had previously used the division symbolism for partition problems did not perceive the situation as division, but used the multiplication sign instead. Although many children used the multiplication sign, they said they thought it was "adding up," showing a connection to the repeated addition model.

Sub-grouping

The representations for the subgrouping problem revealed a range of intuitive strategies and invented symbolism which showed variations of the structure of division.

Some children needed to check that halving 6 apples would give 12 pieces and matched "12 people to 12 pieces" in a one-to-many correspondence process. Figure 19 shows how Bianca represented the apples "halved" and matched one person to each half accordingly. She visualised and explained the halves as separate for each person without drawing them again. As well, she correctly symbolised the process $6/12 = \frac{1}{2}$ even though she had not been instructed in the use of the division symbol or related part/whole division situations. Catherine used a similar approach and drew the halves matched to each person. She symbolised the process as "12 halves makes 6", but was not aware of any relation to multiplication.

Figure 9.20 shows her use of invented symbolism to specify that she meant "12 lots of 1/2" rather than "12 and one half." In comparison, Michelle thought that "you could not divide 12 apples between 6 people and that 6 people would have to without, or they would have to eat oranges."

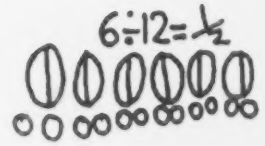

Figure 9.19: Subgrouping (Bianca)

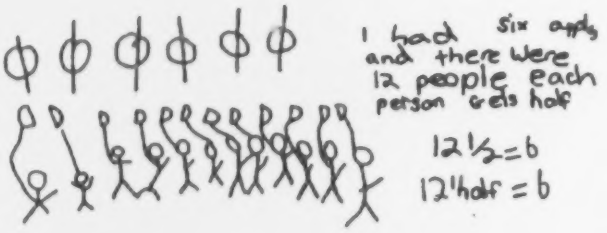

Figure 9.20: Subgrouping (Catherine)

For the large number subgrouping problem involving 6/24 most children represented the process as repeated addition after halving and re-halving mentally. Those children who guessed quarters actually found that 12 halves makes 6 (based on the earlier problem), and then doubled 12. This was performed efficiently and with an almost spontaneous sense of knowing that the answer would have to be one quarter. This may have occurred because children relied on their experience of making quarters. Another reason may be that the number of apples was far less than the number of people and children estimated that "little bits would have to be quarters" (Rebecca). Natalie used the counters to represent 24 and halved and re-halved as a check. Figure 9.21 shows how Rebecca symbolised this process as 6 + 6 + 12 = 24 as well as 4 x 6 = 24.

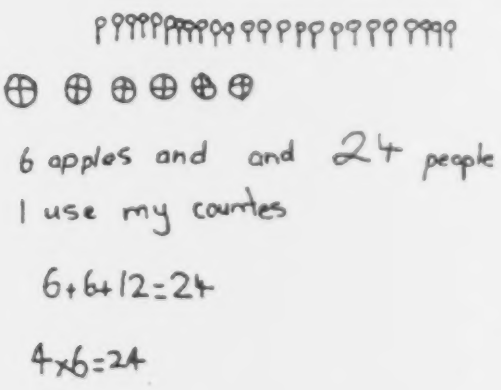

Figure 9.21: Subgrouping (Natalie)

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Multiplication facts using the multiplication sign, and addition symbolism and procedures were prevalent, but in some cases the division sign was used correctly. It was apparent that some of the children had already been exposed to division through a range of experiences outside the formal teaching of mathematics at school.

Conclusions

This study showed that with minimal guidance, young children are able to represent their solution strategies to a range of multiplication and division situations without prior classroom instruction. Children did not need to master computational skills in order to solve these problems, nor did they require any particular training in problem-solving and representational strategies. They worked independently and co-operatively, explained and justified their representations, and expressed their problem-solving strategies in writing. Focusing on children's representations of mathematical situations presented a wealth of information about the child's conceptual development of multiplication and division: their counting and number fact knowledge, their developing intuitive models of multiplication and division, and related problem-solving and representational strategies.

Existing intuitive models for multiplication and division (direct counting, repeated addition and subtraction, and multiplication operation) were clearly identified, and calculation strategies were identical to those found in the longitudinal study (Mulligan & Mitchelmore, in press). However, because children were encouraged to show their strategies in a variety of ways, their range of representations often showed strategy use of varying sophistication. In some situations, children may have chosen to represent situations and structure diagrams in a way that causes the least "cognitive strain" (Bell, Greer, Grimison & Mangan, 1989). What we need to identify here is evidence that the child has actually developed multiplication and division as an operation, and whether their calculations and recordings reflect this. The pictorial recording always showed the semantic structure of the problem, but students sometimes re-structured it so it was similar to other situations they could calculate easily. Although the pictorial recording showed the semantic structure of the situation, the child's calculation strategy, and their explanation, actually provided evidence of the intuitive model used. The multiplication and division notation did not necessarily mean that the child was using an operation model of multiplication; it may be misleading in that multiplication symbolisation can be used as a summary of repeated addition and have no meaning as an binary operation in itself.

The intuitive models used by the children did not necessarily correspond to the semantic structure of problems. There was a strong preference for the repeated addition model across all semantic structures. In contrast to Fischbein, Deri, Nello & Merino (1989), the representations did not did not show that children solved equivalent groups, partition and quotition problems

using models only reflecting these semantic structures. Children used a set of calculation strategies that they applied to all problems.

It appears that children's active processing of mental images is important in developing conceptual understanding of multiplication and division. Images are built up from models of situations, semantics, notations and calculations, and these can be facilitated through representations that are described, drawn, compared and discussed. As internal structures of multiplication and division are developing children's external representations may show invented symbolism and unconventional details. The examples shown in this chapter revealed each child's unique internal constructions of the multiplicative situations at that time. Evoking a range of representations is, in our view, positive and supports the growth of intuitive models. Identifying a child's range of intuitive models is critical if we want to challenge the child to operate with the most sophisticated strategy available to them and to develop this. Thus the teaching/learning situation needs to provide opportunities for children to develop and represent meaningful representations of problems and their solutions.

Limitations

The conclusions are of course limited in a number of ways. This was a descriptive study and it was not designed or intended as a controlled experiment permitting immediate generalisation. The problems used in this study were restricted, and although we included a wider range of semantic structures than previous studies, there are still many others which could be investigated. Our methods of inferring aspects of children's internal representations from their externally produced representations are still exploratory. Children may select from a range of available representations for recording or elaboration or it may not be possible for a child to record their internal representations accurately. In some cases, children may have been persuaded by their peers to represent a problem in a particular form.

The length of time of the study was also limited. A more detailed picture of each child's developing structure of multiplication and division could be ascertained through longitudinal analysis of case studies, by monitoring children's development over several years.

Implications for teaching

Overall, the findings of this study suggest that by providing appropriate experiences to encourage mathematical representations children can be supported to acquire more sophisticated and wide-ranging calculation and problem-solving strategies. The challenge for teachers is to recognise and interpret children's informal mathematical knowledge and to assist children to build upon and represent their existing knowledge structures.

The widespread practice of teachers imposing on children, pictorial models or notational structures that represent contrived mathematical representations, often presented in traditional

textbooks, needs to be seriously questioned (Clark & Kamii, 1996; Thompson, 1992). Imposed and contrived representations of multiplication and division may restrict or prevent the child from forming and using meaningful representations. In turn this may impede the development of conceptual knowledge that is linked to the child's image. It would be desirable then to develop instructional programs that provide children with appropriate experiences in representing a range of problem-solving situations that at the same time, enhance conceptual understanding and the development of new mathematical ideas.

Longitudinal studies would assist in gaining a clearer picture of how representations of multiplication and division can promote conceptual understanding. It may be advantageous then to encourage children's representations and problem solving in the early development of multiplication and division.

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Chapter 10

Children's Drawings of Multiplicative Structures: Cartesian Product and Area

Lynne Outhred

Introduction

It has been found that children faced with their first arithmetical problems use intuitive mathematical schemes to solve them and often persist with these methods despite teachers' explanations of conventional approaches (Steffe, 1994). Insights into children's intuitive methods can be gained by asking children to explain the methods they use, together with observing models or drawings they make to help them reach solutions.

It is fascinating to watch children as they make drawings to solve or to explain their solutions to problems. Drawings indicate the variety of approaches and may also highlight the aspects, sometimes irrelevant or unexpected, that influence children's intuitive strategies. Why do drawings seem to be so informative as a means of understanding children's thinking? Perhaps because children must provide the structure for their drawings, so they seem to invent their own methods rather than trying to reproduce a standard form. Moreover, young children's drawings may provide more information than they can express verbally. The focus of this chapter is children's drawings as indicators of their understandings of one concept: multiplication.

This chapter begins with an overview of the structure of multiplication. Drawings of two particular problem types, cartesian product and rectangular area, obtained from the author's research are then examined in detail. Finally the drawn solutions are compared with a system of representational structures for multiplicative situations developed by Behr, Harel, Post and Lesh (1994).

The structure of multiplication

Units

The operations children first learn about are addition and subtraction, in which the units are individual entities and the same unit is used throughout. For example, counters and pencils are not added together unless the objects are related to a common referent such as "things". Schwartz (1988) refers to these operations as *referent-preserving*.

As children move from additive to multiplicative situations with whole numbers, there are two important unit-related changes (Hiebert & Behr, 1988). Firstly, new, more complex units are introduced. These units might be multiple or composite entities. Children gradually realise that objects that have been counted can be considered as new units that can be counted further. For example, a collection of marbles may be considered as a number of bags (each with the same number of marbles) or a class of children might be thought of as a number of pairs, groups or car-loads. This shift from single entities as units to constructions of composite or multiple entities as units is crucial to the development of multiplication (Steffe, 1992). Such a shift is not trivial as it represents a fundamental change in numerical understanding; "a change in the nature of the unit is a change in the most basic entity of arithmetic" (Hiebert & Behr, 1988, p. 2). However, these units still refer to amounts that can be quantified by counting or measuring.

The second change is that multiplication almost always involves the manipulation of relationships between quantities, for example, apples per box, price per kg, or km per hour. Schwartz (1988) draws attention to the complexity of these referents. He argues that multiplication is a *referent-transforming* operation, since two quantities with different referents are taken as input, and the output is a quantity whose referent is different from either of the first two. For example, "You have three bags of oranges with ten oranges in each bag. How many oranges do you have?" requires an answer as "oranges", not as "bags" or "oranges per bag". The quantity "ten oranges per bag" provides no information about the number of oranges or bags, only the relationship between them.

Situations in which units are continuous rather than discrete, and fractional rather than whole, provide further challenges to children's understanding of the nature of units. Both quantities in the preceding problem about oranges are discrete, whereas in a problem such as "The bus travelled at an average speed of 56.5 km/hour for 3.25 hours. How far did it travel?", the quantities are continuous and a repeated addition model is not applicable. Schwartz (1988) points out that the notion of multiplication as repeated addition is problematic because addition is a referent-preserving operation while multiplication is not.

Being able to anticipate the unit structure of the solution may be necessary before children can attempt many multiplication problems. Both cartesian product problems and area

problems involve changes in the type of output unit, and this change may be difficult for children to grasp. The idea of a unit as a combination of two quantities (e.g. fillings and bread types combining to give different choices of sandwiches) may not be obvious. Similarly, children may not be able to imagine rectangular area as a tessellation of units of a particular shape and size, the number of which can be calculated from the lengths of the sides. To understand such complex situations, children have to build up networks of relationships among quantities (Thompson, 1994). To build up networks of relationships among quantities, knowledge of problem structure rather than calculation procedures is required. Children's drawings may indicate the nature of their knowledge of such relationships.

Multiplication structures

Vergnaud (1983) classified multiplicative problems and situations into structures for which concepts, procedures and representations of different but inter-connected types are necessary. He suggested that knowledge of the different multiplicative structures is complex and gradually develops over many years. He postulated three different types of multiplicative structures. The first two structures are described here; the third is beyond the scope of this chapter and will not be discussed.

- **Isomorphism of measures**

  The "isomorphism of measures" structure comprises situations where there is a simple direct proportion between two quantities. Such situations are exemplified by asymmetric problems in which multiplier and multiplicand have distinct roles, such as "4 pencils at 10 cents each", which can be represented as a direct mapping between the set of numbers of pencils and the set of costs. Such problems can be solved by multiplication (4 x 10) or repeated addition (10+10+... (4 times)). Vergnaud noted that the alternative repeated addition procedure, 4+4+4... (10 times), is not constructed by young children because it is not meaningful, a property that has been described as "psychologically non-commutative" (Mangan, 1989).

- **Product of measures**

  The "product of measures" structure includes problems where two quantities are combined into a third. In this structure there is a double proportion (the initial two quantities with the resultant quantity).

  Two multiplicative situations that fit into the "product of measures" structure are cartesian product and rectangular area problems. Multiplication in these cases is both symmetric and psychologically commutative. For example, in rectangular area problems the product of length x width is conceptually the same as width x length, although the two forms of the solution are not perceptually identical (unless length equals width). Similarly in cartesian product problems, two quantities (e.g. the number of blouses and the number of skirts) are transformed into outfits which represent combinations of these. Neither blouses nor skirts is clearly distinguished as multiplier or multiplicand. Vergnaud believes the arithmetical structures of both area and cartesian product

problems are difficult because such structures cannot be mastered until they can be analysed as double proportions.

Students have been found to be more successful on simple, rather than multiple proportion problems (Vergnaud, 1992). However, Vershaffel, De Corte and Van Collie (cited in Nesher, 1992) found that varying the quantities from integral numbers to fractions and decimals increased problem difficulty for asymmetric problems. This did not occur for symmetric problems, suggesting that different models may be used for the different problem structures.

Repeated addition is only natural in the *isomorphism of measures* structure, where one quantity is replicated a number of times. Since this model constrains one dimension to be integral, it does not generalise to more complex situations. Initial teaching of multiplication is usually based on a repeated addition model, perhaps because this model readily links to physically modelling situations using concrete material. This emphasis on repeated addition has been challenged by Confrey (1994, p. 1) who claims that "models of multiplication based on counting and/or repeated addition do not adequately explain many of the actions of young children which can be seen as multiplicative", in particular, actions such as sharing, folding, dividing symmetrically, and magnifying. She asserts that treating multiplication as predominantly a succinct version of repeated addition, in which the roles of multiplier and multiplicand are carefully distinguished, often results in an inadequate understanding of multiplication as a flexible, binary operation involving scaling, magnification or growth.

What models do young children associate with multiplication and what strategies do they use to solve multiplicative problems? English (1982) interviewed twenty children in Year 2 and determined the models they used when solving simple number sentences. The majority of the children used the equivalent set rather than the array model, but the reason for this preference was not discussed. The area model (based on rectangular arrays) seems to be even less well known than the array model in which elements are not contiguous. Although both models have a row and column structure children may not see the structural similarity between the two. Beattys and Mayer (1989) found that although some children were familiar with an array model (lines of people), these children did not associate the area model (a rectangular tiling of six rows, each of four squares) with multiplication:

> Because it doesn't say nothing about multiplication. It doesn't give you no explanation on it and it just.....it's only blocks with squares and you don't know what it's telling you.
> This picture of like some squares....this picture would come out to like a times table chart....you'd fill in the numbers on it.

If children do not perceive that discrete arrays of objects and area models (in the form of grids) have the same structure, their concepts of multiplication will be limited. While the array model is applicable to many multiplicative problems in which quantities are discrete, an

understanding of area requires children to realise that a rectangular tessellation is also a representation of multiplication. If children do not connect the two different models, they are unlikely to understand why areas and volumes can be calculated from linear dimensions.

Array and area models can effectively demonstrate the commutativity of multiplication, and are often used in the teaching of fraction and area concepts. These models can also serve to link different forms of multiplication, such as repeated addition and cartesian product. Children's lack of familiarity with array-based models of multiplication has important implications for teaching as the equivalent set model becomes increasingly cumbersome for multiplication of large numbers, does not illustrate commutativity, does not generalise to non-integral quantities, and does not play an important part in the development of higher-order concepts.

Representations of product of measures structures

An important aspect of solving a problem is translating it into a form that is meaningful to the solver, perhaps using concrete materials to model the problem, making a drawing, creating a mental image, or rewriting the problem. For example, the starting point for word problems is the text and the end product is usually a numerical solution. Nesher (1992) points out that the translation process is not simply a mapping from one language to another. It seems to require identification of key elements and relationships in a linguistically coded problem. Moreover, young children may first need to translate a written or verbal problem into a form that assists them to count using, for example, their fingers, counters, or drawings. In particular, drawings may indicate children's awareness of problem structures; enable children to check their solutions; act as links between concrete and symbolic representations and aid the development of counting skills.

The following sections describe the results of a study which investigated the representations and strategies used by 115 children from Grades 1 to 4 at four Sydney schools as they solved a sequence of tasks (Outhred, 1993; 1995). The tasks included finding the area of a rectangle and three word problems. Children were encouraged to draw solutions to the tasks and the methods used were monitored, that is, whether the children drew a picture, counted on their fingers, pointed to imagined units, or counted "in their heads". For the word problems, drawings were produced by children for two different purposes: one to model the problem as an aid to counting, the other to explain the solution process to the interviewer. These drawings often indicated whether children saw the problem in terms of enumeration of individual units, repeated addition or as a product. The sequence of tasks related to children's early understandings of area included covering rectangles using cardboard squares and drawing arrays of squares. These tasks were designed to investigate children's intuitive notions of area as the covering of a region with identical units and their understandings of the key properties of

array and area models, namely, that the units are congruent and aligned vertically and horizontally with equal numbers of units in each row (column).

Cartesian product and area problems are both *product of measures* structures. What do the drawings tell us about children's understandings of this structure?

Cartesian product

Cartesian product problems are complex because they involve the notion of categories rather than individual elements that can be directly represented. The cartesian product problem that the children were given was:

> A shop sells ice cream cones. There are four flavours of ice-cream (chocolate, vanilla, strawberry and banana). The cones come in three sizes (small, medium and big). How many different ice-cream cones can you choose?

The other two problems would be classified as *isomorphism of measures* structures: an equivalent set problem (groups of children) and an array problem (lines of children).

The difficulty levels of the equivalent group problem and the array problem were almost identical (they were correctly solved by 54% and 53% of children respectively) whereas the cartesian product problem was far harder (21% of the children successfully solved it). Few children solved the latter problem before Grade 4; one child in Grade 1, four in each of Grades 2 and 3 and almost 60% of children in Grade 4 gave correct solutions. Of the children who were unsuccessful in solving this problem, a quarter of the children did not attempt it or focused on a specific aspect (e.g. which flavours they would like) while half of them repeated one quantity (the number of flavours or sizes) or added the quantities. Younger children considered the numbers given in the problem to be actual objects so "three" was the number of cones, rather than the number of cone sizes. As a consequence they tried to map the quantities in a one-to-one correspondence (e.g. large strawberry, medium banana and small chocolate with some vanilla ice-cream left over). The comments children made reflected their confusion: "One flavour wouldn't have a cone," or, "But there's only three sizes." Only two children gave incorrect multiplicative solutions.

Incorrect solutions to the cartesian product problem may have been less prevalent if the children had been given "hands-on" problems, such as those used by English (1993) to investigate children's problem solving strategies. In her study children dressed toy bears in different combinations of outfits. However, she did find that many children were reluctant to select an item more than once, even though multiple items were available. She attributed children's reluctance to their interpretation of the problem wording; her tentative explanation was that children interpret "different" to mean "different in all ways", rather than "all the different outfits". Her explanation does not seem sufficient to account for children's difficulties with this problem structure. In the study by Outhred (1993) if the child repeated one quantity,

the interviewer gave a hint, either "Is a small chocolate ice cream the same as a large chocolate one?" (if the child's answer was the number of flavours) or "Is a small chocolate ice cream the same as a small strawberry one?" (if the child's answer was the number of cone sizes). The usual effect of a hint was a change from focusing on one quantity (say, cone sizes) to the other quantity (flavours) or to an additive solution. Very few children gained any insights into the problem structure. An exception was Michael (Grade 2) who had interpreted the problem to mean the number of different flavours, but the hint made him revise his interpretation of the problem.

Michael	Four. [Hint given] Eight, because there's big and medium and small so there's more.
Interviewer:	Could you draw the cones and show me.
Michael:	[Draws twelve cones] There's two meanings. I thought you meant just these [indicates four flavours] but you could mean all these. Which do you mean?

The hint only seemed to be effective if children could conceive of quantities as categories rather than individual entities. Once children realised that each quantity could be repeatedly combined with the other the difficulty was to systematically enumerate all possible combinations.

The drawings of the children who successfully solved the cartesian product problem could be classified as unsystematic, systematic but not an array, and systematic array (see Figure 10.1). There did seem to be a relationship between increasingly structured drawings and correct solutions. Such a relationship may be explained by the increased knowledge of multiplication that was shown by the children who solved this problem. In general, these children drew representations to explain their solutions rather than to work out the answer. Thus, the representations that they drew were more likely to show groupings which clearly linked to array models of multiplication than those of children who drew the ice creams as a means of keeping track of all the combinations.

The different types of representations children drew were strongly influenced by the problem wording and context. The use of the term "lines" in the array item prompted children to draw linear structures whereas the use of the term "groups" (related to a classroom context) in the equivalent set problem resulted in less structured drawings.

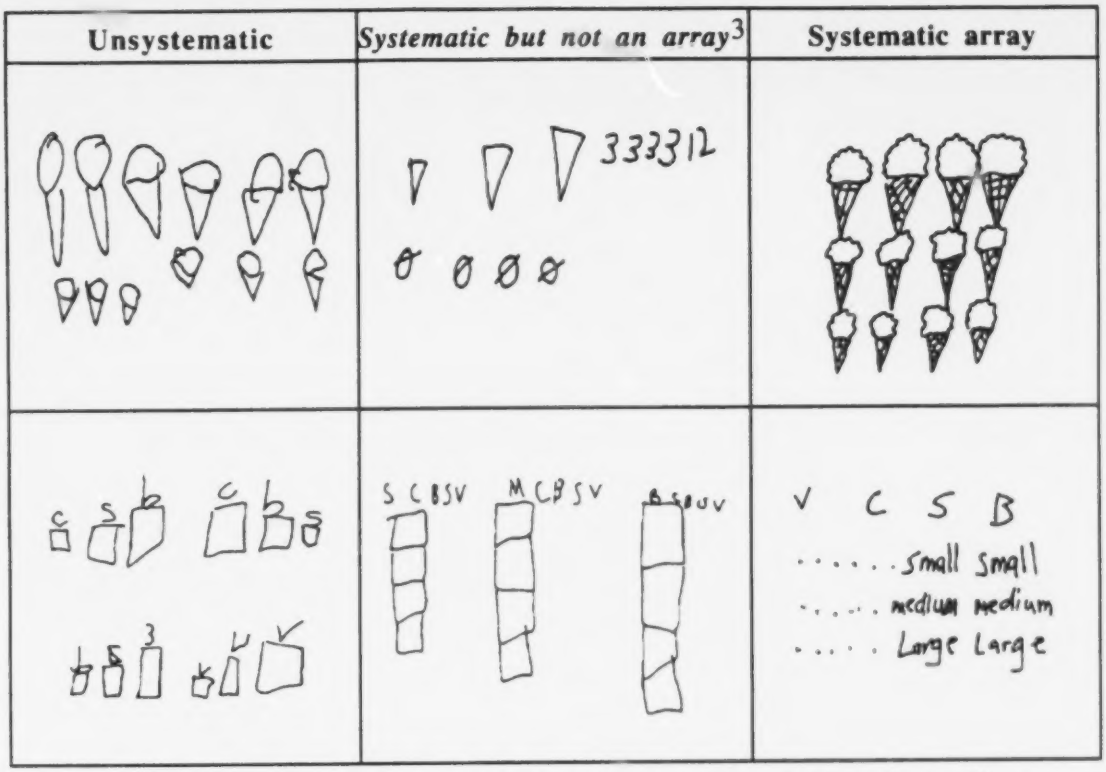


Figure 10.1: Examples of the successful children's drawings for the cartesian product problem

Rectangular Area

Unit conversion (e.g. measuring the length of a bookshelf in centimetres instead of in metres) does not change the nature of the referent but only the numerical description of its measure. However, for a measure such of the area of a rectangle, the multiplication operation relates two different quantities, length and area. Thus, to understand area, children have to grasp the nature of area units and the reason that these are measured in square units (cm^2, m^2, etc.) rather than in length units (cm, m, etc.). Schwartz (1988) comments that:

> Normally, we think of area as a two dimensional construct. But thought of in this way, it is not possible in any consistent way to order a set of given areas. ... It is only when one dimensionalises the product space, making a one-dimensional construct whose measure is cm^2 (in this case) that one can answer ordering questions unambiguously. (p. 51)

To understand how children link the structural features of arrays to multiplication, it is necessary to consider the strategies children apply to multiplicative situations that involve arrays and how these strategies develop in the early years of school. Bishop (1988) comments that "the realisation that an array can be drawn as two perpendicular sets of parallel lines, opens the

[3] The calculation in the first example shows repeated addition of 3 + 3 + 3 + 3 = 12. The cones were systematically counted for each flavour and the flavour then crossed out.

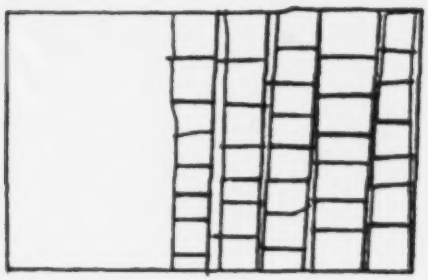
knowledge and allows the student to see the link to multiplication and side length relatively easily". Other researchers have observed that children who successfully solve area estimation problems frequently talk about imagining lines or strips across the figures; for example, "squares formed when the up and down strips met the back and forth strips" (Hirstein, Lamb & Osborne, 1978). These strategies relate to visual representations and may not be apparent when children cover areas with concrete materials as the covering is usually constructed by placing the units individually.

Figure 10.2 illustrates the difficulty that many children in Grades 1 to 4 had in seeing groups of elements in an array and in representing the relationship between the rows and columns (Outhred, 1993).

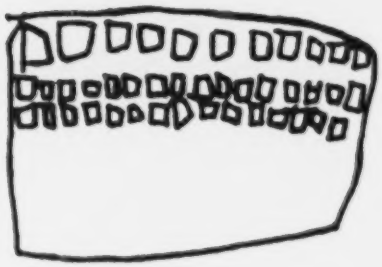
The drawings show examples of responses to the task of copying a 3 x 4 array that the children had made with cardboard squares. It seemed quite hard for children, particularly those in Grades 1 and 2, to maintain both the alignment and the numerical structure in their drawings, even though the concrete model of the array was in front of them. Some focused on the total number of squares and drew twelve squares with no row or column structure, and some drew a rectangle and covered it with "squares" without maintaining the numerical structure (as the first drawing in Figure 10.2 shows). The majority of children (70%) drew an array of the correct dimensions, but about 50% of the children in each year drew each square individually. Such a strategy was not successful when children were asked to draw larger arrays, where their concerns with covering the whole rectangle influenced their judgements about the size, alignment and number of units. Many of the Grade 3 and 4 children changed their strategy to drawing lines, a strategy that was not available to most Grade 1 and 2 children.

Similar difficulties were apparent in the responses to another task in which Grade 1 and 2 children were shown a large array (14 x 18), told the number of squares along its left and top sides (these were pointed out on the figure), then asked how many squares there would be on the bottom side. Some children realised without counting that there would be 18 but others counted to determine how many. However, not all children who answered "18" without counting were confident that there would be the same number in each row. Thus, when asked how many there would be across a row in the middle of the array, some children answered that they did not know or counted the number of squares in the row. These results indicate that young children may simply not realise that aligned units of arrays have the same number of units in each row (column).

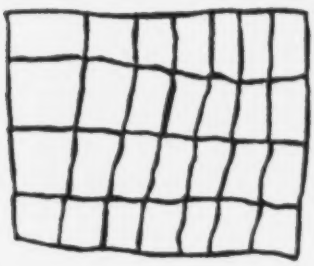
Incorrect number of columns with an unequal number of squares in each column



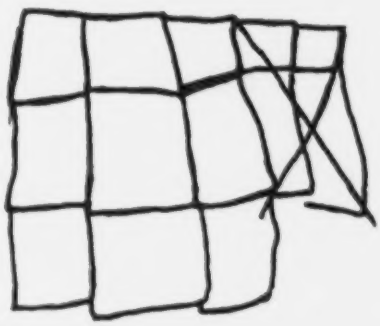
Correct number of rows with an unequal number of squares in each row



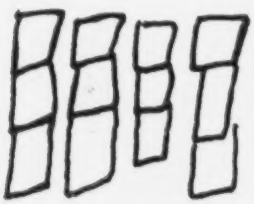
Incorrect number of rows with an equal, but incorrect, number of squares in each row



Correct number of rows with an equal, but incorrect, number of squares in each row



Correct number of columns with an equal and correct number of squares in each column



Correct numbers of rows and columns

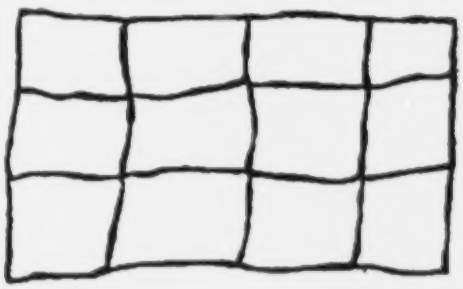


Figure 10.2: Illustrations of children's drawings when copying a 3 x 4 array.

Rational number construct theory analysis

Some researchers (Behr et al., 1994) believe that children's responses are essentially task dependent and have hypothesised that cognitive structures based on these responses will also be task dependent. Therefore, research findings about cognitive functioning will depend on the researchers' knowledge of the mathematical concepts involved and on the tasks that are used. Behr et al. propose that researchers need to develop detailed analyses of the mathematical constructs being investigated and to formulate hypotheses concerning the associated cognitive structures necessary to understand these constructs. They analyse the mathematical constructs that they hypothesise "represent knowledge structures needed to represent a cross product of two quantities with a manipulative aid or with standard notation" (p. 142-143) and recommend that children should be helped to develop these intuitive knowledge structures through carefully designed experiences. In particular, these authors have suggested that *product of measures* multiplication (which they call "cross-product" multiplication) involves being able to reunitise quantities in different ways. They present a sequence of hypothesised reunitisations for both cartesian product and rectangular area problems to exemplify discrete and continuous situations respectively.

In this section the theoretical sequences presented by Behr et al. (1994) will be interpreted in relation to children's drawn responses to two specific examples, the ice cream problem (see above) and finding the number of square centimetres that would cover a rectangle (Outhred, 1993). The purpose of the analysis is to determine if their theoretical sequences provides a framework for classifying children's drawings in terms of developmental levels. If their sequences are evident in the drawings of children who successfully solved these problems, then their argument that these aspects of children's cognitive structures need to be developed would be supported.

Cartesian product

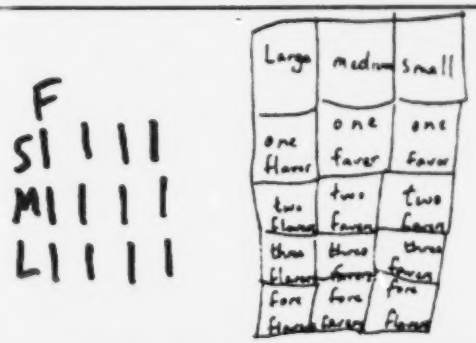
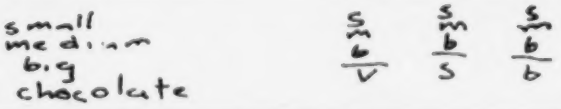
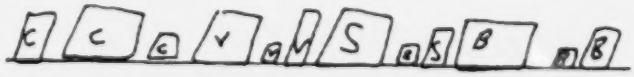
Table 10.1 shows the drawings of the children who successfully solved the cartesian product problem matched to each of the theoretical reunitisations postulated by Behr et al. (1994, p. 141). An obvious notation (invented by the author as a simplification of the original notation) is used to represent the reunitisations: The size unit is represented by SMB and the flavour unit by CVSB. The size unit decomposed into singleton units is represented as (S+M+B) and the flavour unit by (C+V+S+B).

Behr et al. suggest that children who can successfully solve the cartesian product problem reformulate the given quantities in the sequence shown by steps A to E. Table 10.1 shows that children's drawings matched their sequence quite well. It must be emphasised that most representations shown in Table 10.1, in particular, those in steps A to C (array-based representations) were provided in response to a request to explain, rather than to work out, the solution. However, as explanations these drawings suggest how each child has conceived the relationships between the quantities. The non array-based representations match the "inner"

reunitisations (steps D and E) in which at least one quantity was decomposed whereas the array representations are more structured, in that the two quantities are seen as related. Only the representations shown in step A can be said to encompass the entire sequence of reunitisations. Representations E to A thus show an increase in children's structural understanding.

Table 10.1

Theoretical analysis and successful children's representations of the cartesian product problem

Sequence of reunitisations (Behr et al., 1994)	**Examples of children's drawings (Outhred, 1993)**
A The quantities were cone sizes (small, medium, large) and flavours (chocolate, vanilla, strawberry, banana). Children were asked for the number of different choices of ice-cream cones that could be made (given only one flavour on a cone): SML x CVSB	Some children immediately multiplied the three cone sizes by the four flavours and wrote this solution as 3x4 giving a verbal explanation such as, "You would multiply the number of cone sizes by the number of flavours because each flavour can come in small, medium and large."
B Reunitise one quantity (either sizes or flavours): (S+M+L) x CVSB or SML x (C+V+S+B)	 The above represent reunitisations of size with flavour remaining composite.
C Reunitise second quantity (flavours or sizes): (S+M+L) x (C+V+S+B)	
D Decompose one quantity (e.g. flavours): (S+M+L) x C + (S+M+L) x V + (S+M+L) x S + (S+M+L) x B	
E Decompose second quantity: SC + MC + BC + SV + MV + BV + SS + MS + BS + SB + MB + BB	A few children drew each flavour and cone size systematically, e.g. small chocolate, medium chocolate, large chocolate, small vanilla, medium vanilla etc. 

The correspondence between drawings and calculation strategy is fairly good. Children who used a repeated addition or a count-all strategy to solve the problem generally drew non-array representations (steps D and E) and all but one of the children who multiplied drew an array structure (steps A, B, and C). This correspondence suggests a build-up from count-all to repeated addition of singleton units and finally to multiplication. The drawings suggest that children who multiplied were indeed able to reunitise the quantities postulated by Behr et al.

An instructional sequence corresponding to steps E to A (in that order) may be effective in helping children to build up an understanding of combinatorial structures in a way which enables them to analyse cartesian product situations as a double proportion.

Rectangular area

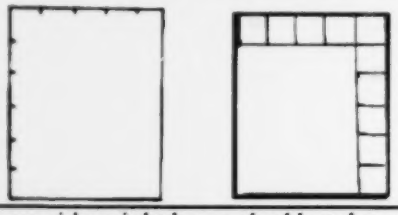
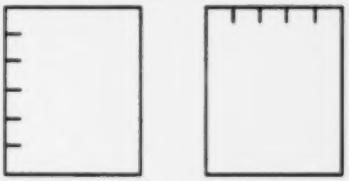
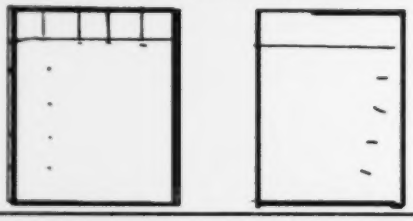
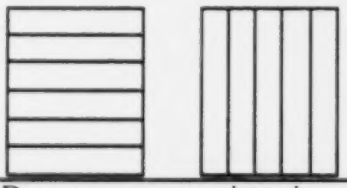
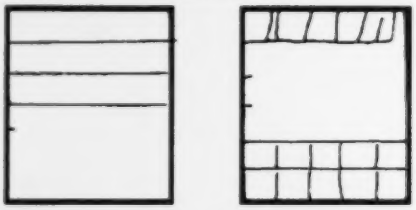
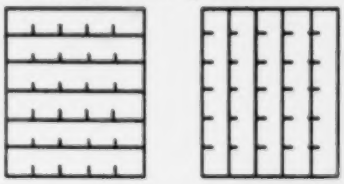
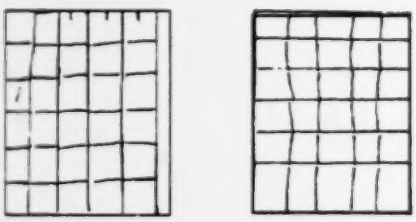
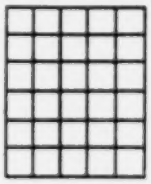
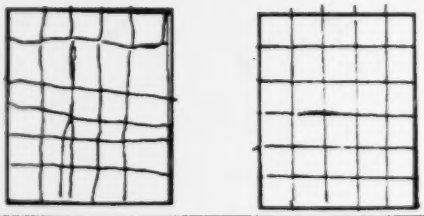
In the area task (Outhred, 1993), children were first asked to measure a line 10 cm long with a ruler. They were then asked to work out how many 1 cm squares would be needed to cover a 6 cm x 5 cm rectangle (one square and the rectangle were shown). Success on this task clearly depends on children's knowledge of centimetre units and array structure as well as multiplication. The results for the area measurement task were very similar to the cartesian product problem. Overall, 24% of the children successfully solved this task; the majority of them (70%) were in Grade 4.

Table 10.2 shows drawings of the children who successfully solved the rectangular area problem matched to each of the theoretical reunitisations postulated by Behr et al. (1994, p. 145) While some drawings matched Behr et al.'s steps, the matching process was complicated by visualisations that were not recorded by the child; in particular it was difficult to distinguish between steps B and C. For example, children often slid the ruler down the rectangle; it was not clear whether they counted rows as composite groups of five, or as five separate units along the ruler because the drawings do not show differences between these strategies. This difficulty was more evident for the area problem than the cartesian product problem because the drawings shown in Table 10.2 were made to obtain a solution, not to explain it. (The drawings that were produced as explanations were easier to interpret because the children usually verbalised their solutions as they drew.)

Again, the "inner" reunitisations correspond to the most primitive calculation strategy, with children's understanding of numerical structure becoming gradually more sophisticated from step E to step A. The calculation strategies followed the same sequence as in the cartesian product problem. Children who made drawings similar to those shown in steps D and E usually used a count-all strategy with steps B and C involving a mix of count-all and repeated addition strategies. In step A, most children used multiplication.

Table 10.2

Theoretical analysis and successful children's representations of the rectangular area problem

Sequence of reunitisations (Behr et al., 1994)	Examples of children's drawings (Outhred, 1993)
A The rectangle is seen as a product unit:	Both dimensions measured or marked off in unit lengths, most children multiplied. 
B Reunitise length or width: 	Only one side might be marked but the other had to be measured or estimated[4]. 
C Treat reunitised quantity as a composite unit: 	
D Decompose composite unit: 	D Rows shown by lines, squares marked off individually. 
E Construct singleton units: 	E Squares are drawn individually. 

[4] In the second example one child drew the top row of squares and marked the left hand column, then counted all the units in each row (1, 2, 3, 4, 5; ...) using the left hand column as a marker.

The most primitive calculation strategy, however, was not necessarily used with the most primitive measurement strategy. Observation of children as they attempted to solve this task suggested that measurement was the most difficult aspect of unit construction. A few children realised that multiplication of the side lengths would give the number of squares and these children did not draw solutions. As the unit was shown the children did not have to anticipate its size and shape. Most children realised that covering required a tessellation of the unit squares. The difficulty was to draw the tessellation of the correct size. Many children who were unsuccessful in solving the area problem could not even draw a correct tessellation.

The evidence of the children's drawings provides little support for Behr et al.'s suggested sequence of cognitive structures for area. Their theoretical sequence does not seem to capture the complexity of the constructions children drew to work out the area of a rectangle. While the drawings E to A seem to show increasing structural understanding, the matching seems to be basically superficial. For example, it could be argued that the drawing shown in E provides evidence of a higher level of understanding of area than D (in terms of both array structure and measurement) because the child has seen the relationship between area and side length. Moreover, one child who drew a structure such as the one shown in A (with squares marked along adjacent sides) used a count-all calculation strategy.

In their sequence, Behr et al. have focused on the subdivision of an area into numerical units (e.g. whether there were five singleton units or a composite unit of five) but have neglected the measurement aspect. How do children actually construct the units? The children who successfully solved this problem already knew the structure of the tessellation and did not need to draw all of it. Children's basic problems were (a) learning the array structure; (b) realising that there is a connection between side length and area units; and (c) partitioning lengths into the correct number of units. If children realised the relationship between side length and area and could partition lengths into the correct numbers of units, then they could solve the area problem without knowledge of multiplication.

Moreover, Behr et al. suggest that their sequence applies to *product of measures* tasks when concrete materials are used. But different concrete materials would seem to add another layer of complexity. Doig, Cheeseman & Lindsay (1995) found that the children's responses to covering tasks using concrete materials were strongly influenced by the materials themselves. When children used paper squares, more information was gleaned about their conceptual development of area than when they used either wooden tiles or Dienes blocks. Children's difficulties in covering rectangles with paper squares were similar to the difficulties they experienced in drawing; the units had to be aligned both vertically and horizontally, with no gaps or overlap. The results of Outhred (1993) would support the conclusion that the materials provided affect children's solution strategies. When concrete materials (cardboard squares) were supplied few children had difficulty working out how many were needed to cover a rectangle. By contrast, constructing area units using a ruler was far more difficult and gave far more information about children's concepts of area than covering a rectangle with cardboard squares.

Although the drawings indicate that the sequence proposed by Behr et al. is inadequate as an explanation of children's understandings of area, an instructional sequence based on steps E to A may assist children to build up their knowledge of the numerical structure of a tessellation. However, children may not recognise the structure to be multiplicative, nor be able to construct the unit tessellation.

Conclusions

This chapter has given an indication of the complexities in representing multiplicative situations that fall into Vergnaud's *product of measures* structure. Children's representations of cartesian product and rectangular area problems indicate why the concepts underlying multiplication are hard for students to grasp. The arithmetical operations of addition, subtraction and division can be relatively easily related to their corresponding physical actions: addition to the result of augmenting a set or joining two (or more) sets, subtraction to removing elements from a set, and division to an equal partitioning of a set. Multiplication, however, is not easy to relate to a corresponding physical action, so it is commonly modelled as repeated addition. The closest analogy to a physical operation is a concept of multiplication as an iteration of a one-to-one correspondence between several sets. While young children can generate one-to-one and one-to-many correspondences, if they do not perceive a whole set as resulting from the iteration of such correspondences, they may not be aware that a situation is multiplicative (Nantais & Herscovics, 1989).

The analysis presented by Behr et al. (1994) is a step towards hypothesising the cognitive structures underlying an understanding of multiplication as a product of two quantities. Their sequences appear to give a logical analysis of the ways in which children might conceive and build up their understanding of *product of measure* structures. However, such sequences need to be verified in work with children as it is difficult to foresee the factors that affect the development of children's multiplicative concepts in specific contexts. The results presented in this chapter would appear to support their hypothesised sequence of reunitisations for cartesian product situations, but not for rectangular area. These results also suggest that children's solution strategies are substantially task dependent and support Behr et al.'s comment that responses to particular tasks may not indicate stable cognitive structures. It was also evident in Outhred (1993) that the solutions children gave depended not only on the specific task, but also on the sequence of prior tasks. Learning experiences that assist children to build up networks of different multiplicative structures and to link these together might help children progress from understandings of multiplication from enumeration of individual objects, to repeated addition and finally, to a product. Analyses such as those postulated by Behr et al. may provide guidelines for the design of such learning experiences, particularly if they are validated by analyses of children's responses.

Asking children to draw representations may help them to focus on features that are relevant to different problems. Many of the children included unnecessary details in their drawings, but when they were encouraged to use more abstract representations (sticks or circles) many were happy to do so. Vergnaud (1990, p. 28) suggests that a major problem in education comes from the fact that "many teachers take mathematical concepts as ready-made objects, without seeing that these concepts have to be built by students as functional tools that will enable them to deal with several kinds of situations". Drawing representations for carefully designed learning experiences may assist children to identify, select and articulate relevant objects and relations. Moreover, explaining their drawings to the class or a group may help elucidate the concepts for other children, as many of the drawings showed clear links between concrete and symbolic representations. In conclusion, drawings are an under-utilised source of information, not only of children's understandings of problem structures but also the sequence in which these structures seem to develop.

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Part 5

Computation and Number Sense

Chapter 11

Redefining Early Number Concepts through Calculator Use

Kaye Stacey and Susie Groves

Introduction

For well over a decade, calculators have been recognised as having the potential to profoundly change the curriculum and the nature of mathematics teaching, with widespread agreement amongst mathematics educators in many countries that calculators should be integrated into the core mathematics curriculum (Cockcroft, 1982; Curriculum Development Centre and Australian Association of Mathematics Teachers, 1987; National Council of Teachers of Mathematics, 1980).

The *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) recommends that all students use calculators at all year levels (K–12) and that calculators be used both as instructional aids and as learning tools. In line with world-wide trends, the *National Statement* emphasises developing number sense through mental computation and estimation. This emphasis is at least partly due to a reassessment of the balance between mental calculation, pencil-and-paper calculation and the use of a calculator that best suits the requirements of work and everyday living in modern society.

However, despite official endorsement, there has been no widespread move towards the use of calculators in primary school mathematics. It is no trivial matter to use technology effectively in classrooms: it is now clear that teachers need to rethink mathematics and children's learning of mathematics, as well as to develop new (and substantially different) skills for teaching and assessment.

This chapter will report on some of the findings of a long term research project into the learning and teaching of mathematics, conducted in six schools where children were given free access to calculators from Kindergarten onwards, and discuss the implications of these findings for the learning and teaching of number in the early years of school.

The *Calculators in Primary Mathematics* project

The *Calculators in Primary Mathematics* project was a four year research project investigating the effects of the introduction of calculators on the learning and teaching of primary mathematics. The project commenced at Kindergarten (the first year of formal schooling in Victoria) and Grade 1 levels in six schools in 1990 and moved up through the school to Grade 4 by 1993. New children who entered Kindergarten joined the project each year until 1992. From 1990 to 1993, a total of 79 teachers and approximately 1000 Kindergarten to Grade 4 children participated in the project at these six schools, with approximately half of the children participating for four years.

Children were introduced to calculators on entry to school (i.e. at age 4 or 5), with all children having access to their own calculator in class. The purpose of introducing calculators was to provide children with a rich mathematical environment to explore, not to encourage them to rely on calculators for computations. The project did not supply teachers with activities or a program to follow. Instead, teachers were regarded as part of the research team investigating the ways in which calculators could be incorporated into their mathematics classes alongside traditional teaching practices such as the use of concrete materials. Teachers were provided with feedback and support through regular classroom visits by members of the project team and were encouraged to share the activities they devised through school and network meetings and the project newsletter. The collection of project newsletters, which contain many of the activities devised by teachers, has been published (Groves, Cheeseman, Dale & Dornau, 1994).

Across the world, there have been other projects which have similarly investigated the effects of calculator use on the learning and teaching of primary mathematics, but most of these have been concerned with upper primary school, have been small scale projects in terms of the number of children and teachers involved or the duration of the project, have not had the use of calculators as the major focus, or have mainly been concerned with the production of classroom materials. A notable exception was the *Calculator Aware Number Project* (CAN), which commenced in Britain in 1986. The CAN project, which worked with children aged from 6 years, began to explore what a curriculum looks like if it takes seriously the implications of the availability of calculators. Although there was no formal evaluation of the learning outcomes for students, the CAN project reported that children developed a wide range of strategies for carrying out calculations and reached a high level of numeracy for their age (Duffin, 1989; Shuard, Walsh, Goodwin & Worcester, 1991; Shuard, 1992; Thompson, 1994). The CAN

project was particularly concerned with examining the impact that calculators had on primary mathematics by freeing children from the necessity to learn written algorithms, with participating schools having to agree not to teach formal written algorithms.
Calculators in Primary Mathematics was a research project which worked with children from school entry to Grade 4. It set out to build on the findings of the CAN project. Unlike the CAN project, teachers and schools were not required to abandon the teaching of standard written algorithms, which, in any case, are not typically taught in schools in Victoria until Grade 3 or later. One goal of the research was to identify the future role of pencil-and-paper calculations in classrooms where there was free access to calculators.

Specific aims of the project were to:

- document the extent to which teachers incorporated calculators into their teaching and the ways in which calculators were used;
- ascertain whether teachers' expectations of children's mathematical performance changed as a result of the introduction of calculators and what long-term effect this had on the curriculum;
- determine the long-term learning outcomes for children involved in the project; and
- explore changes in teachers' beliefs and practice in regard to the learning and teaching of mathematics.

Data was collected through regular classroom observations; teacher interviews; written teacher evaluation sheets and questionnaires; and written tests and interviews with children. A large scale survey of teachers was also undertaken to determine attitudes and practice related to calculator use in the wider community.

A major focus for the project was the way in which calculators can help children construct ideas of number and build number sense. Among the original hypotheses of the project was the expectation that project children would successfully deal with larger numbers and acquire certain concepts (such as place value, negative numbers and understanding of decimal notation) at an earlier age than other children.

All Grade 3 and 4 children in the project schools in 1991, 1992 and 1993 (approximately 500 in each year) were given a written test and a test of calculator use, while over 10% of these children were individually interviewed for about 25 minutes. The 1991 children, together with the 1992 Grade 4 children (none of whom had been part of the calculator project and could therefore be assumed to have had minimal exposure to calculators) formed the control group for the study; the other children were involved in the project for approximately 3 years.

An analysis of the Grade 3 and 4 interviews showed that children with long-term experience of calculators performed better than children without such experience on a range of computation and estimation tasks and some "real world" problems; exhibited better knowledge

of number, particularly place value, decimals and negative numbers; made more appropriate choices of calculating device; and were better able to interpret their answers when using calculators, especially where knowledge of decimal notation or large numbers was required. Written testing, with and without calculators, confirmed that these children had better understanding of the number system and were somewhat more able to identify an appropriate operation in a word problem. No detrimental effects of calculator use were identified. For further details see Welsh (1992); Groves (1993, 1994a, 1994b); Stacey (1994a, 1994b); Stacey and Groves (1994).

This chapter will report those findings which are concerned with Kindergarten to Grade 2 children's construction of number concepts and their teachers' consequent reconstructions of their classroom practice to accommodate their changing views of what constitutes appropriate content and pedagogy.

Using calculators with young children: attitudes and practice

In 1990, a survey of 700 teachers at 100 Melbourne schools was conducted to determine teachers' attitudes towards calculator use and the extent and purpose of calculator use in their classrooms. Results indicated a remarkable shift in favour of the early introduction of calculators since a similar survey was carried out ten years earlier by Ferres (1981). In the 1990 survey, 75% of teachers supported calculator use in Kindergarten to Grade 3, compared with a mere 7% in 1980. These attitudes, however, did not necessarily translate into practice. In the 1990 sample, 58% of K-3 teachers admitted to rarely or never using calculators in their classrooms.

It might be hypothesised that the main reason for such a low rate of calculator use was their lack of availability. However, at the time of the survey most primary classrooms in Australia boasted a computer costing approximately ten times the price of a class set of calculators. Any lack of availability of calculators in K - 3 classrooms must reflect the priority attached to their use, rather than their cost.

The introduction of calculators to young children requires a realisation by teachers that their use is not restricted to finding answers to computational exercises and an awareness of other uses which might be incorporated into their mathematics programs. It is significant that while teachers in 1990 expressed a high level of support for "creative" uses of calculators, this was not matched by their teaching practice. The gap was particularly evident for K - 3 teachers. For example, over 80% of teachers at all year levels supported the use of calculators to allow for more challenging problems solving situations and also to develop new concepts and skills. Yet in both instances, 60% of K - 3 teachers admitted to rarely or never using calculators for these purposes.

Therefore, a first step towards realising the full potential of the calculator for young children was a demonstration of ways in which it can be used for purposes other than computation.

The major ways calculators were used in project classrooms

During 1990 and 1991, a large body of anecdotal data was collected through an extensive program of classroom visits by the project team and by teachers' self-reporting of activities. Teachers were visited once a fortnight on average and completed approximately one record sheet per month. Four major ways of using the calculator emerged, as described and illustrated below.

The calculator as a recording device

Young children spontaneously used their calculators as a "scratch pad" to record their telephone numbers, their scores when playing games, interesting patterns and even the date. Writing numerals can be a time-consuming and onerous task for young children, so calculators provide the opportunity to record (often very large) numbers easily and change them at will.

Teachers exploited this spontaneous use of the calculator as a recording device by devising many activities which simply used calculators to record numbers. For example, a teacher of Grade 1 and 2 children devised an ordering activity, *number line-up*, which began with a small group of children entering numbers of their choice into their calculators and then sorting themselves into ascending order according to the numbers on their calculator display. More and more children were added to the line-up, with each new child needing to find their correct position. Children frequently included negative or very large numbers in this activity and exhibited a quite sophisticated knowledge of the number system.

The teacher who devised this activity used it and other similar activities over and over again. She commented that the activities she now tries to use are ones where the children can "take themselves where they want to go" and that she is never sure what is going to happen mathematically. She identified this as a major change in her own teaching of mathematics and believes that she now uses many more open-ended activities than in the past.

The calculator as a counting device

Project teachers found that one of the most effective uses of the calculator with young children is as a counting device. The built-in constant function on the calculator allows counting by any chosen number, from any desired starting point. For example, if a child enters 2 + 3 =

= = ... , the calculator "counts" by 3's, giving 5, 8, 11, 14, ... By watching the display, children are supported in learning to count – they can match the numerals to the words for small numbers, see what number comes before or after any given number, learn about place value by counting by tens, and make discoveries such as how to count by odd numbers on their calculator or how to count backwards (e.g. counting backwards from 10 in steps of 1 can be achieved by entering 10 – 1 = = = ...).

One Kindergarten teacher initiated an activity which she called *number rolls*, which became popular with many project teachers. Long strips of paper were used to vertically record counting on by a constant. Many children began by counting by 1's and continued to do so. Others, however, moved on to counting by numbers such as 5, 10 or 100. At least one child observed that counting by 9's usually leads to the units digit decreasing by one each time while the tens digit increases by one. This activity continued over many weeks. In the classroom where it started, just as the teacher was ready to abandon the activity, she noticed that many children were beginning to make conscious predictions about the next number in their sequence–even when they could not necessarily read the numbers aloud. In another Kindergarten class, Ben had counted up to 17 900 by 100's on his number roll. When asked what number would be reached after pressing equals two more times, he wrote 18 100, although he read it as "eighteen hundred and one". In yet another Kindergarten class, Daniel wanted to show how he could count by 50's to 4000 on his calculator. Before he could be stopped, he had reached 64 250, which he was able to read aloud without hesitation.

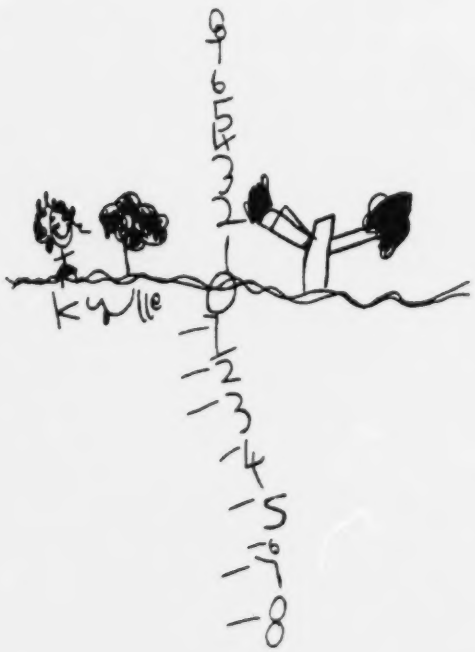

Figure 11.1: Kylie's underground numbers

By using the constant function to count backwards, children discovered and explored negative numbers. In a Kindergarten class where children had been discussing and drawing "What lives underground?", Alistair said "Minus means you are going underground" and wrote some negative numbers. When questioned what would be the first number above the ground, he said "zero". "Underground numbers" were used and discussed freely in many classrooms. Kylie's illustration is shown in Figure 11.1.

A Grade 2 child, Sebastian, after playing a game of stepping stones on the classroom floor, went away to make his own stepping stone numbers. His teacher had used whole number steps, with numbers in the range 0 to 25. Children had to step along, counting the numbered stepping stones as they went, while following instructions such as "go up three steps", "go back six steps". Sebastian also began his steps at 0. However, he chose steps of tenths, reading them as "point 1, point 2, ...". When he reached the step after 0.9 he said "point 9, point 10 – no that's 1". Having made the bridge, he then continued easily. A problem arose when Sebastian began to use his stepping stones in an actual game because the operations he chose (go up 3 steps, go back 6 steps, etc) took him below zero. He stopped for a moment to consider what would happen if the numbers "hadn't been points". Soon he was producing stepping stones for counting backwards into negative decimal numbers.

The calculator as a computational tool

Calculators were sometimes used to solve real-life problems, where the size of the numbers would otherwise have been prohibitive, or to carry out numerical investigations. An example illustrating how the presence of the calculator allows children to work with larger numbers and solve more realistic problems was the *tree survey*. Grade 1 and 2 children worked in groups to tally the number of small, medium and large trees in the large, heavily treed schoolground. Children used several different methods to count the trees, including tallying with pencil and paper and keeping count on their calculators. They used their calculators to find the combined totals (up to a hundred for each category). Groups obtained different answers, which led to a thought provoking class discussion about which numbers to use when representing the results in graphical form. Children made considerable progress towards developing an understanding of some quite sophisticated statistical concepts.

While discussing how to make a pictograph of the results of another tree survey, Grade 2 children encountered decimals. A group of 7 children had found 64 trees of a particular category and needed to cut out 64 pictures of trees to paste onto the chart. When asked by the teacher how many trees each child in the group would need to make, children used their calculators and found that 64 ÷ 7 = 9.1428571. The teacher wrote down their answer and asked what it meant. A child quickly replied: "It is nine and a bit. So if we made ten each we would have some left over – actually we would have 6 left over".

On another occasion, Kindergarten children sorted the teddy bears they had brought to school according to colour and counted how many teddy bears there were altogether – some

counted aloud by ones, some by twos, others used the constant function on their calculators, while a few used the calculator to tally the groups as 7 + 4 + 3 + 4 = 18. When they went back to their tables to record what they had done, some children drew bears and others wrote number sentences. Byron immediately wrote the number sentence in the text above and then proceeded to find as many ways as he could to get 18 on his calculator as the sum of other numbers – e.g. 9 + 1 + 8 and, even more remarkably, 6 + 6 + 6. Although the calculator was performing the actual additions for him, he needed to predict which numbers would lead to a sum of 18 and only used the calculator to confirm or contradict his predictions. While Byron used the calculator to confirm that 6 + 6 + 6 = 18, he had observed that this was a likely relationship from the other patterns he had seen, illustrating how the calculator encouraged exploration and provided opportunities for the development of number sense. Byron's work is shown in Figure 11.2.

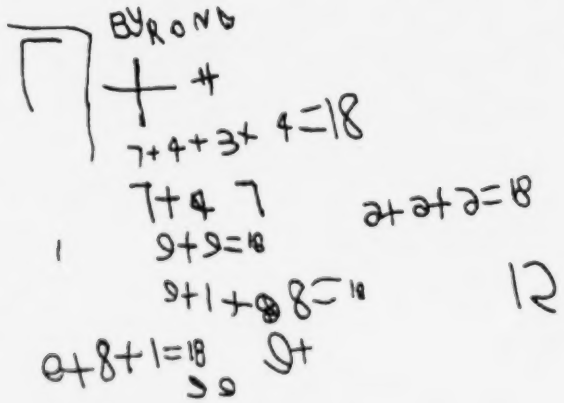

Figure 11.2: Byron's ways of making 18 using his calculator

Other Kindergarten children wanted to share 55 cookies among their 10 teddy bears. One child had discovered the "sharing sign" on the calculator and had informed classmates of her discovery. Two children got the answer 5.5 on their calculator. The teacher discussed the idea that .5 was a half and one child commented "Oh, that's five and a half cookies then". She remembered this and explained it to a member of the project team who visited the classroom some time later, along with the calculation 55 ÷ 10 = 5.5. Figure 11.3 shows the ten bears that Zoe and Julienne pasted onto their paper. They have drawn five and a bit cookies alongside each bear.

Figure 11.3: Sharing cookies among teddies

The calculator as an object to explore

The calculator was also used as an object of discovery in its own right. Children at all levels were keen to discover the functions of the various keys and to establish for themselves that the + key, for example, has the effect of determining "how many altogether". Individual children who discovered how to use certain functions, such as the memory, quickly passed on the information to others who were interested. This was by no means the whole class. Children were fascinated to find out how to switch off their solar-powered calculators – many Kindergarten children tried to see what happened to the display when they put the calculator under their tee-shirts. A class of Grade 2 and 3 children accidentally discovered that the calculator switches itself off when not in use. The teacher used this for an experiment to find out how long it takes. Children devised methods for accurate timing, discussed how many trials would be needed for a sufficient degree of accuracy and found ways to calculate a rough average of the different trials.

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Analysis of calculator use and purpose in project classrooms

On the basis of the teachers' record sheets and notes of the lessons which had been observed during 1990 and 1991, the project team constructed a classroom observation schedule. The observation schedule allowed lesson segments to be coded according to a number of criteria, including:

- the way in which the calculator was being used (i.e. as a recording device, a counting device, a computational tool, an object to explore, or not used at all in the lesson segment observed); and
- the intended purpose for using the calculator (i.e. non-mathematical use, checking answers, routine calculations, teaching or reinforcing concepts, problem solving, problem solving in a real context, or open investigation).

The formal schedule was used in 1992 and 1993 to record, at monthly intervals, observations of mathematics lessons of randomly selected teachers, spread across grade levels and experience in the project. Stacey (1994) analysed the records of all observed lessons by the 11 teachers at three of the schools during 1992. The 101 lessons were broken into natural segments (e.g. a three segment lesson may have begun with an oral whole class activity, been followed by individual work and ended with a class discussion). Segments not using the calculator were discarded. The ways in which calculators were used in the remaining segments were then analysed. Figure 11.4 shows calculator use for the K–2 subset of this sample.

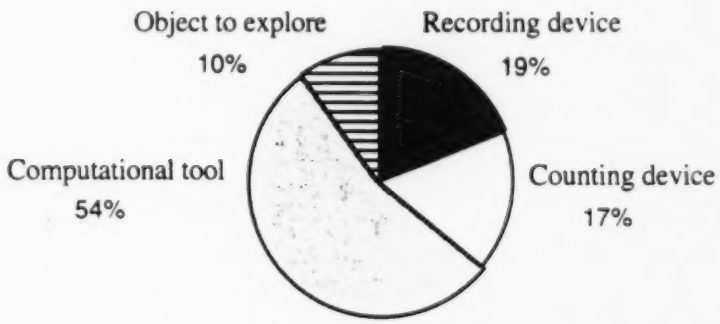

Figure 11.4: Percentages of K–2 lesson segments coded according to each category of calculator function

The intended purposes for using the calculator were analysed in a similar way, after the segments with non-mathematical uses were again discarded. Figure 11.5 shows the intended purpose for using calculators for the K–2 lesson segments.

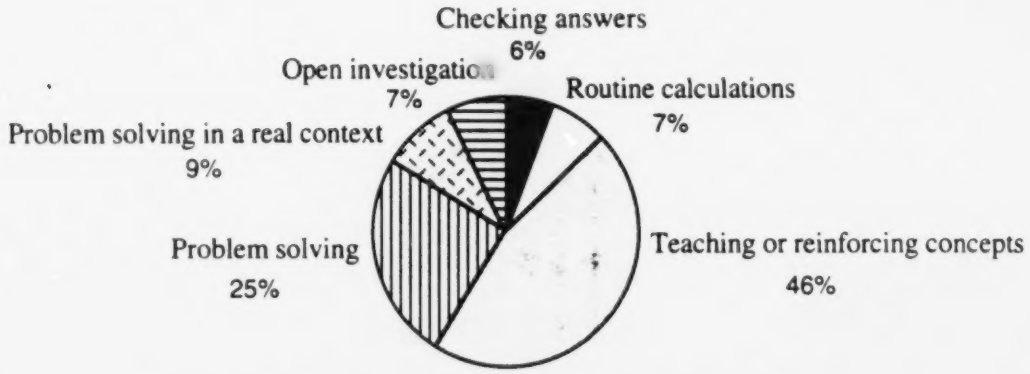

Figure 11.5: Percentages of K–2 lesson segments coded according to each category of intended purpose of calculator use

Although using calculators as a computational tool was the most frequent use of calculators observed in the lesson segments, many of the lesson segments categorised in this way employed imaginative ways of teaching or reinforcing mathematical concepts, the most common intended purpose at all grade levels. For example, in a lesson on inverse operations, children in Grade 2 used their calculators to play *mystery number*. One child entered a mystery number into their calculator and then carried out a sequence of operations suggested by class members – e.g. add on 100 and multiply by 52. The class was then told the answer and had to work out the mystery number. John commented: "I know a number that you can put in to stop anyone finding what your mystery number is – times zero". He explained "Times zero cancels everything and you just end up with the opposite of whatever the last number is to undo."

The calculator was used as a counting device in 17% of the K–2 lesson segments, some of the flavour of these lessons has been described in the section above. Using calculators as a recording device was common in Grades K–2. At the Kindergarten level, about 25% of lesson segments used the calculator as a recording device and it remained a frequent use in lessons up to Grade 2 (about 15%). Using calculators as an object to explore was observed in about 10% of K–2 lesson segments. This often arose when the calculator did something unexpected, such as displaying E when overflow occurred. Children enjoyed finding out why things like this happened and learning to control the events.

Only rarely were calculators used to check answers, with 6% of K- 2 observations falling into this category. In total, checking answers and routine calculations accounted for only 13% of the observations. This is a heartening result in view of the survey results reported earlier, which showed that, while teachers supported non-routine uses of calculators, the majority rarely or never used them in this way.

For further details of classroom activities and the role of the calculator, see Stacey (1994b); Groves, Cheeseman, Clarke and Hawkes (1991); MacDonald (1990); Groves, Ferres, Bergfeld and Salter (1990); the videotape *Young Children Using Calculators* (Groves

& Cheeseman, 1993); a collection of activities used by teachers (Williams, 1992); and the resource book subsequently published by a group of the project teachers (MacDonald, Rogan, & Semken, 1994).

Measuring children's understanding

Classroom observation confirmed the hypothesis that some children were developing concepts related to large numbers, negatives and decimals at an earlier age than expected. In an effort to discover the extent to which children across the whole spectrum were able to engage in these exciting discoveries, interviews were conducted in 1991 with a 20% random sample of children at each of Kindergarten, Grades 1 and 2 at two of the six schools. These interviews were suggested by the project teachers. The first draft of the questions for each grade level were developed by teachers at a project network meeting, without much opportunity to consult among the grade levels or any attempt to make the interviews consistent.

All three interviews, however, included questions where the children were shown large numbers on flashcards and asked, "Can you tell me what this number is?". The percentages of children who could correctly read large numbers (and a decimal for Grade 2) from a flash card are shown in Table 11.1. A significant proportion of the children were able to read numbers which are usually considered well beyond the curriculum at each of the grade levels, as specified by the Victorian *Curriculum and standards framework: Mathematics* (Board of Studies, 1995). Such numbers are indicated by shaded cells in the table.

Table 11.1
Percentages of children able to correctly read given numbers at different grade levels

Level		Numbers presented and % correct responses*			
Kindergarten (n = 29)	Number	74	126	203	
	% correct	46	7	4	
Grade 1 (n = 24)	Number	203	1435	62 750	1 000 000
	% correct	58	46	21	12
Grade 2 (n = 23)	Number	87	372	3294	5.7
	% correct	100	100	65	39

* Shaded table cells indicate numbers beyond the normal curriculum

In addition, 30% of Grade 2 children (the only grade level asked) were able to correctly explain the meaning of –5 as "5 below zero" or similar. Only 9% of Grade 2 children when asked, "How big is 5·7?" were able to give a correct answer such as "a bit bigger than 5".

Thus, many children showed some understanding of negative numbers but, although many could also correctly read a decimal, relatively few could give an indication of its size.

Understanding the meaning of the four operations and being able to select the appropriate ones to deal with real-life problems are essential skills, especially when calculators are used. It was hypothesised that children who had taken part in the project would be better able to select appropriate operations. This was confirmed by the interviews and tests for Grades 3 and 4 children (Groves 1993, 1994b; Stacey 1994b; Stacey & Groves, 1994). In the interviews discussed here, Kindergarten children were asked to tell the interviewer the purpose of the operation keys. The data in Table 11.2 shows that most children in Kindergarten could explain the function of the + key, but not the other operation keys. The = key was widely confused with +.

Table 11.2

Percentages of children able to correctly identify and/or use operations on the calculator

Level		Items and % correct responses*			
Kindergarten (n = 29)	"Explain"	+ key	− key	x key	÷ key
	% correct	68	7	11	32
Grade 1 (n = 24)	"Find"	5+3 (Minties)	**5+27 (minties)**		
	% correct	75	75		
Grade 2 (n = 23)	"Find"	12+3+6 (marbles)	32−13 (dollars)	8 x 5	**20÷5 (balloons)**
	% correct	91	78	73	21

*Shaded table cells indicate numbers beyond the normal curriculum

Grade 1 and 2 children were presented with a number of word problems on flash cards, which were read aloud by the interviewer. They were asked to show how to find the answer with and without the calculator (for example using blocks) and to write a number sentence for the "story". For example, the problem referred to as "5+3 (Minties)" in Table 11.2 was presented as "I had 5 Minties in my pocket. Someone gave me 3 more. How many would I have now?" Flash cards were used in order that children would not need to memorise the numbers involved. Table 11.2 shows the percentage of children who were able to correctly perform the appropriate operations on their calculators.

The results of the interviews help to put the exciting anecdotes of the project into perspective. Many children were operating well above the normal curriculum but the calculator was not creating miracles.

Teachers' expectations

The data presented above offer no comparison between children who had or had not used a calculator. The project did conduct such a comparison at Grade 3 and 4 level, but there was no direct information about changes in performance for K - 2. For this reason, a second mechanism for investigating possible changes in children's understandings was used. This tracked teachers' expectations of their classes. Its validity relies on the fact that almost all project teachers were experienced teachers who had previously taught without calculators. Data on teachers' expectations was collected through interviews and an extensive written questionnaire. The questionnaire contained 121 specific attainment targets, relating to number, at levels of difficulty ranging from what would usually be accepted as relevant to Kindergarten children (e.g. "count aloud to 10") to items which would normally not be seen as part of the primary mathematics curriculum (e.g. "know the answer to 2 – 6 "). To avoid preconceptions about levels of difficulty, the items were presented to the teachers in a random order.

At the beginning of each year of their involvement in the project, teachers were asked to indicate, for each of the items, whether they expected all, most, some or none of the children to be successful on the item at the beginning of the school year and at the end of the year. Each teacher's own completed questionnaire was returned at the end of the year for them to indicate, in the same manner, children's actual performance by the end of the year. At the end of the year, teachers were also asked to comment freely, in writing, on whether and how their expectations had changed.

A total of eleven items on the questionnaire related to what could be termed large numbers at the K–2 level (e.g. "correctly read 14 560", " know the value of the 2 in 521 400", "write in figures four thousand and seventy three" and "count in 10's from 960 to 1050"). Four items on the questionnaire related to negative numbers (e.g. "count backwards in 10's from 50 to -70" and "know that -3 is the number immediately to the left of -2 on a number line marked off in 1's"). A further six items (after the deletion of one item which was clearly misinterpreted by teachers) related to decimal notation (e.g. "know which is bigger, 0.75 or 0.8", "know that $^1/_3 = 0{\cdot}3333333$ on a calculator" and "count from 0.1 to 1.1 in 0.1's").

Groves and Cheeseman (1992) analysed the 35 questionnaires for K–2 teachers in 1990 and 1991 for the items relating to large numbers, negatives and decimal notation. For the purpose of analysis, the responses relating to the number of children expected or actually able to perform a task was mapped onto a linear numerical scale as follows: None = 0; Some = 2; Most = 2; All = 3. For each item, the mean response for teachers at each grade level was then calculated, as was the difference between the actual performance (as rated by teachers at the end of the year) and that which the teachers had predicted for children at the end of the year. These differences indicate the extent to which more (or fewer) children were actually able to do the tasks than had been anticipated by their teachers.

Table 11.3 summarises these results by showing the mean of these differences across each of the three types of items for each grade level. None of the entries in the table are negative. This shows that children equalled or outperformed teachers' expectations in all three areas – this was particularly true of Kindergarten and Grade 1 children for large numbers and negatives.

Table 11.3
Mean differences between children's actual performance and teachers' expectations on various types of items by grade level

		Mean difference*		
Type of item	No. of items	K (n=13)	Gr 1 (n=12)	Gr 2 (n=10)
Large numbers	11	0.3	0.3	0.1
Negative numbers	4	0.2	0.3	0.1
Decimal notation	6	0.0	0.1	0.2

*Mean difference between teachers' reports of children's actual performance and their expectations on various types of items, using the scale of None = 0 ; Some = 1 ; Most = 2 ; All = 3 .

The summary of the data in Table 11.3 conceals the overwhelmingly positive differences by item. Across the total of 21 items and three grade levels discussed here, the mean difference for each item at each grade level was negative only twice (at the grade 2 level with –0.1 for "correctly read numerals to 1000" and –0.2 for "count in 10's from 960 to 1050"). A mean difference of zero was recorded for a further 12 cases (four of these being for decimal notation at the Kindergarten level), while the remaining 49 cases were all positive.

The two items which showed the greatest difference between teachers' expectations and children's actual performance were "count from 389 to 407" at the Kindergarten level and "understand what is meant by -7 on a calculator display" at Grade 1. For the latter item, teachers at each grade level gave an average rating of 1 for actual performance at the end of the year, indicating that on average they felt that some children in every class could read a negative number with understanding.

Differences in the area of decimal notation were smaller than for large and negative numbers, supporting the previous finding that very few children were able to do these tasks. The mean ratings of performance were also lower. Nevertheless, some teachers at all three grade levels reported that at least some children were able to do these tasks. Moreover, at the Grade 2 level the largest differences occurred for the items relating to decimal notation.

In the open-ended comment section of the questionnaire and the interviews, teachers frequently expressed the view that their expectations had changed considerably. One teacher wrote:

> My expectations have changed enormously. It has changed and challenged both the children and myself. There now seems to be no limits to what we can do in our theme work. I am surprised by the children's understanding of concepts – particularly place value, numbers and decimals (including money).

Another commented:

> I feel I have been underestimating the upper reach of some children. Calculators have identified more specifically the range of abilities within the group. I feel children should not be limited to work within a given range of numbers at a given grade level and calculators have certainly confirmed that children can use, understand and be motivated by large numbers.

Curriculum issues

As can be seen from the data reported above, some young children in the calculator project encountered and dealt successfully with numbers that they would not normally be expected to meet until middle or upper primary school, or even junior secondary school. For many teachers one of the frightening aspects of calculator use is the possibility that children may encounter very large numbers, negative numbers and decimals "before they are ready". Project teachers who had become comfortable with calculators in their classroom took the opposite view. They saw their previous curriculum constraints as having imposed artificial boundaries on the children (Groves & Cheeseman, 1992; Groves, 1995).

A Kindergarten and Grade 1 teacher commented that:

> Once ... in Grade 1 we didn't extend beyond 50 at the most when we were counting. The understanding was thought to be limited for children of that age and once they ran out of fingers and toes, that was as much as we expected them to do. But they really do have an understanding now, and they can translate what they are finding out on their calculator into concrete materials.

Nevertheless, while the presence of the calculator appears to allow children to display a knowledge of mathematics which surprises their teachers, many children, especially early in their Kindergarten year, are still struggling with number recognition and are very uncertain about all aspects of the calculator. The wide range of skills and understandings present in any classroom appears to be highlighted by the presence of the calculator. Some teachers report reluctance in making major alterations to the curriculum because they want to cater for the children who they perceive as struggling. The new realisation of the range present in their classrooms presents teachers with difficulties, especially those who feel they must adhere to a rigid curriculum.

Other teachers recognise the changes brought about by the calculator and welcome the way in which the mathematics curriculum has expanded. A Grade 1 and 2 teacher summed up the views held by some teachers that not only has the content of their mathematics changed, but also the extent to which the content is child-centred and more open-ended.

> I'm a lot happier to go where the children want ... it's a lot less teacher directed... The calculators have enabled us to do that ... You don't have to structure things so that you know the answer will be within their reach... If the children want to find out stuff, I say "go for it"... because I'm not so worried about them finding out things they won't understand any more... I used to think what's the point of kids talking about thousands... because they're not going to know what they are talking about ... After having done this for quite a while now I think they *do* understand what they're talking about to a certain extent – enough anyway to use it for their own needs and they will get to understand it as they get older. But why impose artificial boundaries ... and say "you're not allowed to know any numbers beyond 1000"? It's so stupid and that's what we used to do ... I think I'm being a lot more open-ended with their activities... I'm putting more on them to do more finding out. I'm just sort of starting them off. In some ways we are still structure driven when we play games ... but the activities I try and do with them are the ones they can take themselves where they want to go ... You never know what's going to happen.

These changing expectations must, in the long-term, be reflected in changes in the mathematics curriculum.

Implications for the early number curriculum

The awareness of the ways in which children use calculators in their classrooms has stimulated teachers to re-assess their expectations of children. Some teachers clearly attributed the perceived changes to the presence of the calculator, while others, like this Kindergarten and Grade 1 teacher, believe that the calculator has revealed previously hidden abilities:

> The capabilities must have always been there ... we just haven't exposed them to enough. The calculator ... tells me that the children are capable of much more than we expected from them ... It's obvious! We're not creating miracles. The miracles were there all the time.

We believe the movement into new realms of number is happening because the calculator is providing a richer mathematical environment for children to explore than is available through the usual combination of concrete materials and children's own mathematical resources. Calculators confront children and teachers with surprises, such as when the answer

to an apparently simple division results in a long string of decimal places – surprises which initiate inquiry and investigation. As a result, many teachers are using more open-ended approaches in their mathematics teaching and are more willing to let the children direct the course of mathematical activity.

This project poses a number of challenges for teachers and researchers of early number learning. One central issue is what it means to understand number and how children can best develop their understanding. The view of early number learning on which curriculum advice is frequently based (see, for example, Board of Studies, 1995) is that children learn about number slowly and that they need to master one aspect before moving onto others. For example, before moving on to two digit numbers, it is said that children should develop a thorough understanding of natural numbers to ten, evidenced by being able to count out specified numbers of discrete objects, write the numerals correctly and answer questions such as finding the number before or two after a given number. After two digit numbers are studied, students may move onto three digits and the hundreds place value, after which come thousands and beyond. Is this careful progression best for learning? When we have shown a video-tape of a Kindergarten child and his teacher counting to "one million one hundred" (Groves & Cheeseman, 1993), audience members often object "but he doesn't really understand the number 1 000 000!". The implication is that the child and teacher are not engaged in profitable activity in the discussion.

For us this highlights the many different facets there are to "understanding" number and notation. What does it mean to "understand one million"? On the one hand, there is the vague understanding that one million is a big number (maybe even the biggest number!), in contrast to the developed understanding which relates its size precisely to powers of ten. This is also in contrast to the type of understanding of "how big a million is" which enables a person to describe a million grains of sand, a million dollars worth of cars or the time for a million seconds. There is also an "understanding" of the patterns in the digits of numbers written in sequence – how the last digits cycle through from 0 to 9, within a larger cycle of the second (tens) digit from 0 to 9, within a larger cycle, ... This understanding of how the digits reached 1 000 000 in the sequence is again in contrast to a developed understanding from place value. We claim these are all aspects of understanding a million and the child shown on video counting to one million appreciates enough of them for the experience to provide an important building block for his construction of number. We proposed that this child can profitably begin to build ideas about a million at the same time as he is coming to understand numbers like 13, 20 and 79.

Another challenge presented by the calculator in the learning of early number concepts is its apparent conflict with ideas of the primacy of concrete experiences for young children. In many ways calculator use is the most abstract of ways of presenting mathematics, working almost entirely through the formal symbol system without referents. However, we have seen many ways in which teachers have exploited the calculator as a physical aid. For example, children can get a very physical feel for the size of a number such as 450 by pressing the = key

45 times to count up to 450 by 10's and then having to press it 450 times to count back down to 0 by 1's. The late Richard Skemp quoted a teacher as saying that when children learn mathematics through instructional games it was "as though the children's thinking was out there on the table" (Campbell-Jones, 1985). The calculator also has this property–despite its abstraction, it has provided teachers with a window into children's thinking about the mathematics they are doing.

A third challenge to researchers into the learning of early number is the way in which children are compelled to use the formal symbol system of mathematics when using a calculator. In recent years there has been a definite move not to formalise operations and to delay written work until at least Grade 1. Many teachers feel that symbolisation (e.g. writing 2 + 3 = 5) which occurs too early hinders children from building a firm understanding. Use of the calculator encourages early symbolisation of operations, especially of addition. This issue was particularly important at one of the project schools, which had just completed a thorough revision of their mathematics curriculum at the commencement of the project. Along with many other changes, teachers had firmly decided against any written symbolisation at the Kindergarten level. The introduction of the calculator immediately made this impossible to maintain. Concepts such as addition were often treated symbolically alongside concrete materials and the language of "altogether". Teachers immediately recognised that their curriculum was going to need to change yet again and that there would be a flow on to other grade levels as well, as teachers in the lower grades covered areas which had been reserved for higher grades.

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Chapter 12

Children's Problem Posing and Problem-Solving Preferences

Lyn English

Problem posing is well recognised as an important component of the mathematics curriculum and is considered to lie at the heart of mathematical activity (Brown & Walter, 1983, 1993; Kilpatrick, 1987; Moses, Bjork, & Goldenberg, 1990; Silver, 1990, 1994; Silver & Mamona, 1989 a,b). The inclusion of activities in which students generate their own problems, in addition to solving pre-formulated examples, has been strongly recommended by the National Council of Teachers of Mathematics (1989, 1991). By observing the types of problems children generate in different mathematical contexts, we can gain considerable insight into the nature of their mathematical understanding.

Problem posing however, has not received the attention it warrants from the mathematics education community. We know comparatively little about children's ability to create their own problems in both numerical and non-numerical contexts, and the extent to which this ability is linked to their skill in problem solving or to their mathematical knowledge (Silver, 1994). We also have little information on children's problem preferences, that is, on the types of problems children like or do not like to solve, and whether these preferences are reflected in the problems they pose. This would seem an important curriculum issue, given that students' dislike for, or lack of confidence in, solving certain problems has been shown to significantly hinder their performance (McLeod, 1992). Research on children's problem posing is thus in need of considerable attention, especially given the well-documented evidence that young children's creativity and open-mindedness in generating and solving problems dissipates as they progress towards the higher school grades (Campione, Brown, & Connell, 1989; Kilpatrick, 1987; Lester, 1989; Resnick & Resnick, 1992).

This chapter reports on part of a larger study addressing children's problem posing in routine and non-routine mathematical domains. The present discussion is confined to

children's problem posing within the computational domain. In contrast to the substantial literature on children's ability to *solve* computational problems (e.g. Bell, Greer, Grimison, & Mangan, 1989; Bergeron & Herscovics, 1990; Carey, 1991; Carpenter et al., 1993; Carpenter & Moser, 1984; Fuson, 1988, 1992a,b; Greer, 1992; Kouba, 1989; Mulligan, 1992; Riley, Greeno, & Heller, 1983), there is little information on children's ability to pose them. We are thus left with an incomplete account of children's computational facility. Part of this facility requires being able to assign several meanings to the formal symbols (+, -, x, +). Unfortunately, children's school experiences rarely provide them the opportunity to interpret these symbols in a variety of ways; the meanings usually assigned to them are those elementary concepts children are taught first (Fischbein, Deri, Nello, & Marino, 1985; Fuson, 1992; Nesher, 1980; Stern, 1995; Stigler, Fuson, Ham, & Kim, 1986).

These elementary meanings include the change-add-to, combine, and change-take-from situations for addition and subtraction, where the missing end must be found (e.g. "Peter had 9 marbles. He won 4 more. How many marbles does he have now?") Fuson, 1992a. These problem types are first presented to children because they can be easily modelled with concrete materials and there is a clear mapping between the given problem situation and the operation required to solve it (Bergeron & Herscovics, 1990; English & Halford, 1995; Fuson, 1992a). Similarly, children's initial experiences with multiplicative situations include the readily modelled equivalent set problems (e.g. "Four children have 3 balloons each. How many do they have altogether?") and the partitive or sharing problems (e.g. "Jenny has 12 balloons to share amongst 4 children. How many will each receive?"). Children also experience quotitive problems in their early computational activities (e.g. "Tom baked 12 cakes for the fair. He will sell them in boxes of 2. How many boxes will he need?").

In contrast to these elementary problems are those in which there is not a clear mapping between the problem situation and the operation required to find the unknown quantity. For addition and subtraction situations, these include comparison problems with an unknown reference set (e.g. "Sally has 6 goldfish. She has 3 more goldfish than Samantha. How many goldfish does she have?") and comparison and equalise situations with an unknown difference set (e.g. "Sue has 7 cherries. Penny has 4 cherries. How many more cherries does Sue have than Penny?" [comparison problem]; "Sue has 9 marbles. Jenny has 6 marbles. How many more marbles must Jenny win to have as many as Sue?" [equalise problem]).

The more difficult multiplicative situations include scalar (or multiplicative comparison problems, e.g. "Penny has 3 times as many cars as John. How many cars does Penny have?" and the cross product problems (e.g. "Sue has 3 different blouses and 4 different skirts. How many different outfits can she make?"). Other problem types include rectangular area (e.g. "What is the area of a floor of length 4m and width 2m?"), and rate (e.g. "Sam bought 8 stickers for $3.20. If each sticker was the same price, how much did one cost?"). It is beyond the scope of this chapter to address these different problem types in detail; further discussion can be found in English (in press), English and Halford (1985), Fuson, (1992a), Greer (1992), Nesher (1988), and Vergnaud (1988).

Numerous studies have documented the difficulties children experience in solving the more complex problem types (e.g. Cummins, Kintsch, Reusser, & Weimer, 1988; Greer, 1992; Lewis, 1989; Mayer, Lewis, & Hegarty, 1992; Mulligan & Mitchelmore, in press; Nesher, 1992; Stern, 1993). The question remains whether children also have trouble generating these types. That is, when children are presented with the standard symbolic statements (e.g. 23 - 9), do they simply generate elementary problems that map directly onto the structure of the sentence, or do they pose some of the more complex problem types? In other words, do the standard sentences promote the creation of a narrow range of problems, suggesting that children fail to recognise the formal symbolism as representing a variety of situations (Baroody & Standifer, 1993)? Research (e.g. Carey, 1991; Carpenter & Moser, 1984) has shown that children tend to choose a number sentence that directly models the action in a problem, rather than a sentence which represents the arithmetical solution. For example, first-grade children would rarely select the standard subtraction sentence as a representation of an equalise problem; rather, they would choose the sentence, a + . . . = b. This is understandable, given that the subtraction sentence is not supported by the semantic structure of the problem nor by the informal solution strategies young children tend to use.

A previous study (English, in press) had shown that third-grade children have great difficulty in posing a range of problems for the standard addition and subtraction sentences, even after participating in a problem-posing program. It remains to be seen in the present study whether fifth-grade children, drawn from the same schools, show greater diversity in their computational problem posing. Following on from the earlier study, the present investigation examined the extent to which children's number sense and general problem-solving skills facilitate their posing of computational problems. The literature provides little guidance here, with conflicting findings being reported. Silver and Cai (1993), for example, found a strong positive relationship between middle school students' problem-solving and problem-posing abilities, while Silver and Mamona (1989b) found no clear link between these components in their work with middle school mathematics teachers. On the other hand, there seems a clearer link between mathematical competence and problem posing, where students with a strong mathematical knowledge appear better able to generate problems (Ellerton, 1986; Leung, 1993).

As previously noted, a further factor that would seem significant in children's problem posing is their preferences for solving particular types of problems (Brown & Walter, 1993). It may be that children tend to pose problems they like or consider easy to solve and conversely, avoid posing problems they dislike or have difficulty in solving. Research suggests that students favour problems that can be solved readily by a known solution procedure, with little mathematical reasoning (e.g. Lester, Garofalo, & Kroll, 1989; Thompson & Thompson, 1989). Whether or not they show a similar trend in their problem posing remains an issue for investigation.

In sum, this chapter examines the following issues, which were explored as part of a larger study of children's problem posing:

1. What types of problems do fifth-grade children pose for the standard computational sentences? Do they interpret the formal symbolism as representing a range of problem situations?
2. To what extent does (a) strong number sense and (b) competence in novel problem solving facilitate diverse computational problem posing?
3. What is the nature of children's problem-solving preferences and to what extent is this reflected in the types of problems they pose?

Method

Selection of subjects

In order to explore the roles of number sense and problem-solving competence in children's computational problem posing, seven classes of grade 5 children (10 years) were first administered two written, whole class tests. The test of number sense drew upon the important understandings cited by the National Council of Teachers of Mathematics (1989) and by Sowder (1988, 1992). These include understanding cardinal number and the relative magnitudes of number, representing number relationships with manipulatives, recognising unreasonable results for calculations, and being able to recognise and use a variety of computational situations and problem structures. The problem-solving test examined children's ability to solve novel, non-computational problems for which they did not have a readily available method of solution and would thus need to apply a good deal of mathematical thinking and reasoning (Baroody, 1993; Charles & Lester, 1982). Included here were combinatorial and deductive reasoning problems, together with general "process" problems (Charles, Mason, & Gallagher, 1985).

Using the children's responses to these tests, an attempt was made to select subjects who showed different profiles of achievement in the two domains, namely, children who were strong in number but weak in novel problem solving, those who were the reverse of this (i.e. weak in number but strong in problem solving), and those who were competent in both domains. Since there were only a few children who demonstrated the second profile, the present discussion is confined to children in the remaining two categories. The two achievement profiles will hereafter be referred to as WP students (i.e. strong in number but weak in problem solving, n = 28) and SP students (i.e. strong in both number and problem solving, n = 23).

Procedure

The children were individually presented a wide range of problem-posing activities in two, 40-minute sessions conducted on consecutive days. Children's responses to two of the activities are examined here. One activity explored the types of problems children create for the written statements: 29 + 36, 92 - 73, 12 x 4, 56 ÷ 4. Each statement was presented, in turn, on a large card and the child instructed to read the statement aloud. The child was then asked to create (verbally only) an interesting story problem that could be solved by the sentence. It was impressed upon the child that the problem should be an interesting and appealing one that could be given to his/her peers to solve.

The other activity addressed children's problem-solving preferences, as indicated in Table 1. The number of problems was kept to a minimum to avoid overloading the children with information. The problems represented a range of types and did not mirror all of the symbolic statements of the first activity. It was felt this would provide greater insight into the different problem features that contribute to children's like and dislikes in the problem-solving domain. To this end, we presented the child with the four problems listed in Table 12.1, displaying them randomly on cards. The child read each problem aloud and was then presented the questions shown; verbal responses only were sought. It was explained at the outset that the children did not have to actually solve the problems but were to give consideration to how they might tackle them. It was felt this would create a less stressful environment where the child could analyse the problems without worrying about "getting them right."

Table 12.1
Activity Exploring Children's Problem Preferences

Problems Presented to the Child	
1.	Martin sold 9 books and 8 magazines. He sold each book for $3.75 and each magazine for $1.95. How much money did he get altogether for the books and magazines?
2.	Four famous sports people entered a television studio. One was a tennis player, one was a swimmer, one was a golfer, and the other was a chess player. Use the clues to find out who played what sport. *Clues* • Mr Bowler is not good at chess. • Both Mr Big's and Ms Ace's sports involve a ball. • Ms Fish can't swim at all. • Neither Ms Ace nor Ms Fish play tennis.
3.	At the Rainforest school's annual sports day, the gold team scored 568 points. This was 49 points more than the green team scored. How many points did the green team score?
4.	The Lazy Days Icecream Parlor has strawberry icecream, chocolate icecream, and mango icecream. It has green cones and purple cones. The toppings are cherry, banana, and blueberry. How many different icecreams could you buy if each icecream has one flavour, one cone, and one type of topping?
Questions asked of the child	
1.	If you had to solve these problems, which problem would you *best* like to solve? Why?
2.	Which problem would you *least* like to solve? Why?
3.	How would you *change* that problem so that you would *like* to solve it?

Results

Children's problem posing for the symbolic statements

Table 12.2 shows the distribution of the types of problems posed by the children in each achievement profile.

Table 12.2
Percentage distribution of posed *P*problem type for each number sentence by achievement profile

Posed problem type	Achievement profile	
	WP	SP
Addition (29+36)		
Change-add-to	61	57
Combine	32	43
Inappropriate/no response	7	0
Subtraction (92-73)		
Change-take-from	82	74
Separate	10	4
Equalise	0	9
Comparison	4	9
Inappropriate/no response	4	4
Multiplication (12x4)		
Equal groups (set)	50	61
Equal groups (array)	14	22
Multiplicative comparison	4	13
Inappropriate/no response	32	4
Division (56÷4)		
Patrition	75	83
Quotition	4	13
Inappropriate/no response	21	4

Note. WP: strong in number sense but weak in novel problem solving (n = 28)
SP: strong in both number sense and novel problem solving (n = 23)

Most children were able to pose an appropriate problem for the addition and subtraction statements, although they showed little diversity in their problem types. They mostly generated the basic change types, namely, change-add-to and change-take-from, with some children offering combine and separate problems. A separate problem was of the form:

> For the yachting race, Sydney to Brisbane, there were 92 boats. Seventy-three didn't make it to Brisbane. How many made it to Brisbane? (created by Richard, a WP student).

Few equalise and comparison problems were created for the subtraction sentence, these being offered mainly by the SP children. None of the WP children posed an equalise problem. Children's equalise and comparison problems were confined to those with an unknown difference set, such as:

> Robert has 92 toys and Lucy has 73 toys. How many toys does Lucy need to have as many as Robert? (equalise problem created by Thomas, a SP student).

> John has 92 marbles. Sue has 73 marbles. How many more marbles does John have than Sue? (comparison problem created by Lucy, a SP student)

The samples were too small to enable a statistical analysis, however it can be seen from Table 12.2 that there were no major differences between the two achievement profiles in the types of additions and subtraction problems created. However, the SP children did show somewhat greater diversity. It is interesting that these findings reflect those of the third-grade study (English, in press). That is, the children in both grades favoured the basic change types overall, with children who were competent in problem solving displaying greater problem diversity. These trends are revisited in later discussion; consideration is now given to children's responses to the multiplication and division statements.

In contrast to their problem posing for the addition and subtraction statements, the children had some difficulty in actually creating an appropriate problem for the multiplication and division statements (refer Table 12.3). This was particularly so for the WP children. By contrast, only one SP child was unable to pose an appropriate problem for the multiplication statement and likewise for the division statement. Children who experienced difficulty in problem creation either could not generate a problem or else created a problem that did not correspond with the given statement (e.g. "I had 4 children. How many more would I need to make 56?"; created for the division statement by Amy, a WP student).

As shown in Table 12.2, children from both profiles posed problems for the multiplication statement that mainly involved equal groups. Most of these problems referred to sets of items but some featured an array model, such as:

> A farmer had 11 rows of trees and in each row there were 3 trees in each row (sic). How many trees were there altogether? (created by Michelle, an SP student).

A few children (one WP and three SP) created a multiplicative comparison problem, such as:

> I have 12 apples and my friend has 4 times as many apples as me. How many apples does my friend have? (created by Jessica, a SP student).

The children showed even less diversity in their problem creation for the division statement. Partition division was clearly favoured, with only a few quotition examples offered. Problems here included:

> There were 56 lollies and there were 4 people. How many lollies would each person get? – they have to be equal! (created by Thomas, a WP student).

> There were 56 babies in the orphans' home and a number of couples came along and adopted 4 babies each. How many couples adopted babies? (created by Martin, a SP student).

Children's preference for partitive problems is not surprising, given that partition situations occur readily in children's everyday lives. Their response supports the theory of Fischbein, Deri, Nello, and Marino (1985) who consider the partitive model to be the primitive intuitive model for division, with quotition being acquired later through instruction. Other studies lend support here, showing how a large number of subjects of different ages interpret a division statement in terms of partition (e.g. Af Ekenstam & Greger, 1983; Graeber & Tirsoh, 1988).

Children did not pose other problem types that occur frequently in their daily lives, such as a Cartesian product or division involving comparison. It would seem that the children's elementary, even intuitive, interpretation of the symbolic statements, established early in their computational experiences, prevented them from entertaining any other problem situations. On the other hand, children might not pose some of the problems they deal with in their informal, everyday experiences because they do not connect these with their formal school activities (e.g. Hiebert & Carpenter, 1992; Hiebert & Wearne, 1988; Mack, 1995; Resnick & Resnick, 1992). This appeared to be the case in the activity dealing with children's problem preferences.

Children's problem-solving preferences

The children showed some surprising preferences for the four problems listed in Table 12.1. As indicated in Table 12.3, the first problem, a multistep rate example involving money, was rarely chosen as a favourite problem. Only 5 children (3 WP and 2 SP) selected it as the problem they would most like to solve, their reasons being either "I'm good at these," or "They're easy to solve." All six WP children who ranked the problem their least favourite stated, "I don't like money problems" and "I'm no good with money." In contrast, only one of the six SP children who disliked the problem claimed to be incompetent with money problems; the remaining five SP children stated the money problem took too long, involved too many numbers, or that the problem was too easy. When asked how they would transform the problem into one they would like to solve, most children suggested changing the amounts to whole numbers.

Table 12.3
Percentage distribution of most and least favoured problems by achievement profile

Posed problem type	Achievement profile	
	WP	SP
Most Favoured		
1. Multistep rate (money)	11	9
2. Deductive reasoning	43	26
3 Comparison subtraction	28	43
4. Cartesian product	18	22
Least Favoured		
1. Multistep rate (money)	21	26
2. Deductive reasoning	25	17
3 Comparison subtraction	7	13
4. Cartesian product	47	44

Note. WP: strong in number sense but weak in novel problem solving (n = 28)
SP: strong in both number sense and novel problem solving (n = 23)

Interesting contrasts between the two achievement profiles appeared in the children's preferences for problem 2 (deductive reasoning) and problem 3 (comparison subtraction). As indicated in Table 12.3, the WP children favoured problem 2 over all others, while the SP children favoured problem 3. The reasons given by the WP children for liking problem 2 pertained to their interest in deductive reasoning problems and to their enjoyment in solving them. A few children also considered them to be easy. The SP children who favoured this problem expressed similar sentiments, adding that they like deductive problems because

"They're hard and they make you think." Children in both profiles who disliked the deductive problem gave mixed reasons. These included "It's too easy" (4 children), "It's too hard" (4 children), "It takes too long" (2 children), "I'm not good at these" (1 child), and "There's only one method" (1 child).

The 10 SP children who favoured problem 3 (comparison subtraction) tended to do so because they considered it easy to solve ("It is just a subtraction problem.") A couple of these SP children expressed their interest in working with numbers while others said they chose the problem because "It hasn't got many numbers" and "It wouldn't take long to work out." The eight WP children who favoured problem 3 expressed similar sentiments. Interestingly, this problem was rarely identified as the one children would least like to solve. Of the five children (2 WP, 3 SP) who least liked problem 3, three claimed it was too easy and suggested changing it to a multiplication or division case.

Probably the most surprising response is children's dislike of the Cartesian product problem. This was clearly the least favoured of all the problems, despite its real-world context. Thirteen WP and 10 SP children considered this their least favourite problem, with the majority explaining that "It is confusing," "It is too hard," "There are too many things to do," and "It takes too long." The five children from each profile who did favour the problem tended to do so because "It is challenging," and "It is hard and makes you think." Some of these children also stated they enjoyed problems of this nature, partly because "There aren't lots of numbers."

Children's response to the Cartesian product example is even more interesting when we compare it with the deductive reasoning problem. Both problems are set within meaningful contexts, are free of symbols, entail a consideration of several relational ideas, and demand considerable mathematical reasoning (at least for the present sample of children). Yet the WP children favoured the deductive reasoning problem over all others and disliked the Cartesian product most of all. It is clear that both the WP and SP children did not recognise the latter as a case of multiplication and shied away from it because of the number of relational ideas to be considered ("too many things to do"). Invariably, when asked what they would do to transform the problem into one they would like to solve, the children stated, "I would not have as many things, like I'd leave the toppings out." While it cannot be determined whether a less complex case would have yielded more favourable responses, it nevertheless remains that these children did not see the Cartesian product as case of multiplication. This could perhaps explain why none of the children interpreted the previous multiplication sentence in terms of a Cartesian product.

Another finding worth noting is the SP children's selection of the comparison subtraction problem as their favourite problem. They readily recognised it as a case of subtraction (as indicated by their comments) yet rarely posed such a problem for the previous subtraction statement, even though they claimed the given problem was easy. Their selection of a problem which could be readily solved supports the findings of other studies (e.g. Lester et al., 1989). It is surprising however, that the SP children expressed this sentiment, rather

than their WP counterparts who were far less confident in their problem-solving ability. The WP children frequently commented on their perceived lack of problem-solving skill (which was reflected in their initial test results) yet favoured the deductive reasoning problem, perhaps because it appeared more like a puzzle than a problem.

Discussion

While it is not possible to draw specific generalisations from the activities reported here, there are nevertheless some interesting findings worthy of further study. The present sample of children, like their third-grade counterparts, did not display much diversity in their interpretation of the standard symbolic statements even though the importance of creating an "interesting" problem was emphasised. On the other hand, the children were only given one opportunity for problem creation with each statement and perhaps might have shown greater diversity if given a second opportunity. However, this did not happen with the third-grade children who created a second, "different" problem by simply changing their story context (English, in press). It is nevertheless of concern that the children in the present study were all strong in number sense yet displayed limited problem creations. On a more promising note however, the children who were also competent in novel problem solving tended to show greater creativity and were also more adept at actually constructing a problem. On a less positive note, it seems that neither the children's number sense nor their novel problem-solving competence was sufficient to overcome the apparent barriers to divergent computational problem posing. While there may be many such barriers here, the most salient appear to lie in children's problem-solving preferences, in their school activities including the disconnection between these and their informal experiences, and in the intuitive models children create for the four operations.

Although the children demonstrated mixed responses to the problem-preference activity, it would seem that many preferred computational problems that appeared "quick and easy" to solve. Sadly, this appeared particularly so for the competent problem solvers even though they seemed confident in their problem-solving ability. It is not surprising then, that the Cartesian product problem was not liked and was not generated in the previous activity. The children's posing of the simple problem types for the computational statements seemed to reflect their dislike for complex problems.

What is readily apparent from the activities however, is children's lack of exposure to a variety of problem situations within the curriculum and the disconnection between their informal, intuitive mathematical knowledge and their knowledge of formal "school maths;" both of these factors would inhibit their problem-posing efforts. Whereas children's real-world experiences include many situations that extend beyond the basic problem types, it seems they have difficulty in transferring this knowledge to computational activities within the classroom

setting (e.g. Hiebert & Carpenter, 1992; Hiebert & Wearne, 1988; Mack, 1990; Resnick, 1987; Resnick & Resnick, 1992).

The remaining barrier to children's problem-posing efforts may be found in the primitive models theory of Fischbein, Deri, Nello, and Marino (1985). They argue that each basic operation of arithmetic generally remains linked to an "implicit, unconscious, and primitive intuitive model" (p. 4). These models comprise the original meanings that were assigned to the operations and are thus behavioural in nature and meaningful to children. The models are assumed to exert an unconscious influence on children's problem-solving efforts and to be largely responsible for the difficulties children encounter when dealing with computational situations. The constraints these models place on children's performance appear to remain long after a concept has been formalised. Hence if children's concepts of the operations are intuitively tied to the basic situations, then these mental models are likely to place constraints on children's ability to entertain other problem cases. This would apply to their interpretation of the present computational statements whose standard structure maps onto these primitive and enactively meaningful change notions. Children would thus have difficulty in posing problems that did not conform to these intuitive models of the operations.

It is evident that we need to develop children's problem-solving *and* problem-posing skills in dealing with computational situations. We need to broaden the types of problem experiences we present children (Baroody & Standifer, 1993; NCTM, 1989) and in so doing, help them "connect" with school mathematics by encouraging everyday problem posing (Resnick, Bill, Lesgold, & Leer, 1991). This includes capitalising on the informal activities situated in children's daily lives and getting children into the habit of recognising mathematical situations wherever they might be. We also need to encourage children to talk about their problem-solving experiences and to share their ideas on what they consider to be enjoyable and meaningful problems. Finally, we need to provide activities which help children assign multiple meanings to the formal symbols and to associate the standard number sentences with problems that do not directly match their semantic structure.

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Chapter 13

Making Sense of Decimals

Kathryn Irwin

Howard's view of the eminent simplicity of decimal fractions is shared by many adults who find them easy to understand. "The system of decimal fractions is so eminently simple that when it is generally understood will entirely displace the clumsy system of common fractions" (Howard, 1887, cited in Kerslake, 1991). They do not see why many students have so much difficulty learning to use them, and why research into this topic is needed. For example, one reviewer commented that since everybody was going to understand eventually, why investigate students' difficulties? I presume that this reviewer was not aware of how many older students have a very weak grasp of decimal fractions, much less adults who do not know how to operate with them. This is not because these senior students and adults have not been taught decimal fractions. It is more likely that they have not grasped what Howard calls the eminent simplicity of the system.

These older students and adults have probably tried to construct meanings for decimal fractions by building on the basis of what they already know. They have attempted to make sense of decimals in much the same way that young children try to make sense of spoken and written language and the counting numbers. In these domains the wider community provides many chances for learners to match their attempts against common, accepted usage. The community does not provide many such chances for checking students' use of decimal fractions.

This chapter is about students trying to make sense of decimal fractions. The study set out to find what knowledge students already had that could be used in helping them to develop an understanding of decimals. The portion of that research discussed here involved pairs of Form 1 and 2 students (11 and 12 years old) working in pairs to solve problems involving decimals. The study produced some useful statistical results which will only be described briefly. More emphasis will be given to the episodes in which students were trying to make sense of decimal fractions in terms of what they already knew about the number system.

Background

Understanding decimal fractions requires a radical reconstruction of students' concepts about numbers. It is harder for students to grasp than a new aspect of mathematics that can be added onto current knowledge without reorganisation. It is likely that this restructuring is difficult primarily because the symbols involved are ones that students have used successfully in the context of whole numbers and therefore they naturally assume that the understandings and conventions that they have used for whole numbers can be used with decimal fractions. Writers who discussed transfer of meaning from whole numbers to decimal fractions include Brown (1981), Ellis (1995), Resnick, Nesher, Leonard, Magone, Omanson, and Peled (1989), Sackur-Grisvard and Leonard (1985) and Tirosh and Graeber (1990). For example, Ellis (1995) looked specifically at the conditions necessary for students to change their concepts rather than transfer them directly. She found that students could not make this change without feedback, and even with feedback change was not easy.

The wider community does provide feedback on some aspects of mathematics. Ginsburg (e.g. 1977) distinguished those aspects that are seen as belonging to the community which included concepts such as "bigger" and counting. Children learning these will get feedback on whether or not they are right. He described other mathematic as belonging to the school, and little outside feedback is given for topics in this category. Nunes, Schliemann and Carraher (1993) have discussed the relative ease with which people learn mathematics which is supported by their everyday experiences and work, but the difficulty they have with similar concepts taught in school. The importance of providing a rich context for items learned in school is emphasised by Atran and Sperber (1980) who say that "Mathematics does not spontaneously arise irrespective of social context but seems to require a richer and more sustained sequences of experience and instruction in order to flourish than, say, basic grammatical knowledge ..." (p 48). Students do not come to understand by memorising an abstract concept, but, as theorised by Piaget and Inhelder (e.g. 1971), build their own images piece by piece in a way that allows them to remember new information, assimilating it into what they already know.

A purpose of the study to be discussed was to find some contextualised uses of decimal fractions that students did understand, and to watch them construct further understanding in rich contexts.

Method

Subjects

Sixteen students were selected from a combined Form 1 and Form 2 class in one of the lower socio-economic areas of Auckland. The students had the majority of their schooling in New Zealand, had been at their present school for at least one year, and had parental consent to participate. Students in Form 2 had had work on decimal fractions for four years and students in Form 1 had had work with decimal fractions for three years. The teacher was asked to rank the 16 students by their general achievement in mathematics. Students were then paired so that the ranking between students was equivalent. In this school students were used to working together, and the teacher expected that none of the selected students would have difficulty working with one another in pairs. Table 13.1 summarises the sample characteristics.

Table 131
Characteristics of pairs of students

Pair	Rank*	Ages *	Form*	Ethnicity*	Gender
1	1 & 9	12 & 11	2 & 1	Samoan & Maori	f & m
2	2 & 10	12 & 11	2 & 1	Pakeha & Maori	m & m
3	3 & 11	11 & 11	1 & 1	Maori/Pakeha & Samoan	f & f
4	4 & 12	12 & 11	2 & 1	Pakeha & Pakeha	m & m
5	5 & 13	12 & 12	2 & 2	Pakeha & Pakeha	m & m
6	6 & 14	12 & 11	2 & 1	Pakeha & Maori	f & m
7	7 & 15	13 & 12	2 & 2	Maori/Pakeha & Samoan	f & f
8	8 & 16	12 & 11	2 & 1	Maori & Tongan	f & m

* Rank, form, ethnicity by teacher report, age by self report of birth dates.

Procedure and materials

Students were withdrawn from their class in pairs. On the first day they were given a pretest to work on individually. This pretest required them to order a set of decimal fractions between 0 and 2, to estimate the answers to calculations involving decimals, and to give a generalisation about how one could tell about the size of a decimal fraction. Over the next four days they took part in three sessions in which they were given problems to work on together. On the first day these problems dealt with the size or order of numbers, on the second day they dealt with addition or subtraction of numbers which included decimal fractions, and on the third day they dealt with the multiplication or division of numbers including decimal fractions. On the following Monday they were given a posttest similar to the pretest but with some additional

items, which they worked on individually. They were given a similar retention test a month later.

The items on the tests were derived from other research and chosen as ones which discriminated understanding in this age group. The item on ordering was derived from Gelman, Cohen, and Hartnett (1989), Hart, Brown, Kerslake, Kuchemann, and Ruddock, (1985), Hiebert and Tonnessen (1978), Resnick, Nesher, Leonard, Magone, Omanson, and Peled (1989), Sackur-Grisvard and Leonard (1985), and Thipkong and Davis (1991). Calculation items came from Mason & Ruddock (1986). The question about the comparative size of decimals came from Resnick, Nesher, Leonard, Magone, Omanson, & Peled (1989) and an additional item about multiplying and dividing by 10 came from Duncan, Capps, Dolciani, Quast, & Zweng (1980).

The problems were of two types: those for which a context was provided and problems using similar numbers but without a context. An example of a problem in context was:

> $1 New Zealand exchanges for 1.5764 Samoan tala. How much would you get for $10 New Zealand? Estimate, then check it on your calculator.

An example of a similar problem without context was:

> How much is $1\frac{1}{2} \times 10$? Estimate, then check it on your calculator.

All problems were based on four confusions that had been shown by students in a previous part of the study. These confusions were commonly demonstrated in situations for which there was no referent or context for the decimal numbers. The four confusions were:

1. A confusion which leads students to treat whole numbers and decimal fractions are separate units divided by a "decorative dot" (Swan, 1993). Students holding this confusion say that the next number after 2.5 is 3.6.
2. Confusions of place and value. This is most commonly seen in the belief that the size of a number is determined by the number of decimal places (e.g. 0.001>0.1).
3. Confusions about different representations that mean the same thing.
4. Confusions about when a 0 is important in decimal fractions.

Four pairs of students worked on contextual problems, and four on decontextualised problems. The pairs were balanced so that the mean rank of those receiving each type of problem was the same. The mean age and class level of the students was also roughly equal.

For solving the problems all students were given paper, ballpoint pens and a calculator without a fraction key. The pairs that worked with contextual problems could also see Sprite containers of 1.5 litres and 500 ml, a six-pack of 355 ml cans, a glass that would hold 255 ml,

a measuring cup which showed measurements up to 250 ml, and a clipping from the newspaper and a sheet from a bank showing the exchange rate between New Zealand dollars and the currencies of Samoa, Australia and the United States.

The researcher sat with the students while they worked together on problems. She made three types of comment. These were repetition or clarification of the directions; requests for further explanation of their understanding; and prompts to try writing, refer back to something they had done previously, or take a mental break. All problem solving sessions were tape recorded and transcribed for analysis.

Overview of results

There was a significant difference between pretest and the second retention test scores overall ($F(1,12)=14.76$, $p,<.01$); a significant difference between the scores of students ranked in the top and bottom half of the sample, thus confirming the teacher's appraisal ($F(1,12) = 8.66$, $p<.05$); and a significant interaction between the tests and the types of problems ($F(1,12)=5.70$ $p,<.05$). The students who worked on the contextual problems made significantly greater progress between the pretest and the retention test than did the other group.

There was also a significant difference in accuracy between the students' initial and final answers to the problems. All students improved in accuracy between their initial and final answers ($F(1,12=194.27)$, $p<.001$). There were also significant interactions between type of problem and initial and final answers ($F(1,12=6.03)$, $p<.05$), and between type of problem, initial and final answers, and the ranking of students ($F(1,12=10.80)$, $p<.01$). Students given contextual problems were more accurate than students given the decontextualised problems ($F(1,12=6.21)$, $p<.05$).

A third index of the effect of the problem solving was the number of episodes in each transcript that indicated some learning. Both the higher- and lower-ranked peers learned from both conflict resolution and peer teaching in all cases except Pair 8 where only the higher-ranked peer appeared to learn. The higher-ranked peer was more likely to learn from conflict resolution, and the lower-ranked peer was more likely to learn from peer teaching, but learning from both sources occurred for both higher- and lower-ranked peers. Learning appeared to happen more often among students working with contextual problems. The data are summarised in Table 13.2.

Table 13.2
Average number of episodes per person in which learning appeared to take place, by peer ranking and type of problem .

Source of learning	Contextual Problems		Decontextualised problems	
	Higher-ranked peer	Lower-ranked peer	Higher-ranked peer	Lower-ranked peer
Conflict resolution	3.50	1.75	3	0.50
Peer teaching	1.25	3.25	0.75	1.50

Steps on the way to knowing

The statistical results give a general picture of the merit of one type of problem over the other, but they do not say much about the quality of the thinking that students engaged in during the sessions. Analysis of the transcripts revealed many cases where students used their existing concepts to move from an incorrect answer to a correct one.

In most cases seen in the dialogues, students were using their understanding of how counting numbers to understand decimal fractions. They showed varying degrees of doubt about the appropriateness of this transfer. The following section gives examples of appropriate and inappropriate uses of this knowledge, and particularly of the struggle students engaged in to make sense of what was and was not relevant.

Use of number lines and place value

Most children learn in early childhood that counting is involves a fixed sequence of numbers. In school, teachers introduce various diagrams to show this. Number lines are the most common diagram. Place value columns are also used for demonstrating the meaning of larger numbers. In this case students were also familiar with "number tents" (see below) which demonstrate how thousands, millions and higher divisions repeat the use of ones, tens and hundreds.

Although ordering on a number line is appropriate for the counting numbers, its additive construction is less appropriate for decimal fractions where the multiplicative structure of the place value system plays and important role. The common use of ordering as an assessment for understanding of decimals (e.g. Ellis, 1995, Sackur-Grisvard & Leonard, 1985) has a "Gotcha!" quality, in that a large number of children are likely to be caught using whole number logic. The present study also asked students to order several numerals given to different decimal places. The majority of students made errors, as might have been expected. What is interesting is the logic behind the errors that they made.

Examples of this ordering that appeared to follow whole number, number line logic suggested that the length of numeral was an indication of its size. Below was one such ordering:

0 1 2 0.5 0.9 0.01 0.10 0.11 0.100 1.4 1.5 1.43 1.425

This student ordered the numbers by ignoring the decimal point and treating them as whole numbers. However, the student placed decimal numerals that started with a 0 (which several students said "didn't count") before those that started with a 1. The numbers 0.01 and 0.10 were probably placed together because they were regarded as the same (other students read 0.01 as 100).

Other orderings combined some understanding of decimal fractions with number line and place value columns. The clearest demonstration of this came from a student who explained:

> Oh yeah, so it goes 10 9 8 7 6 5 4 3 2 1 and then here's a decimal and then it goes the other way so there's um 1, that's one digit, and then there's 2 digits, which makes it smaller.... After a decimal, say there's one digit there, then it's bigger than a number with say three digits after the decimal, because it's getting closer to the whole numbers.... And this goes point one hundred, point one thousand:
>
> 0
> 1 0 9 8 7 6 5 4 3 2 1 .1
> .10
> .100
> .1000

Consistent with this understanding was the ordering:

2 1 0 0.5 0.9 0.10 0.01 1.4 1.5 1.43

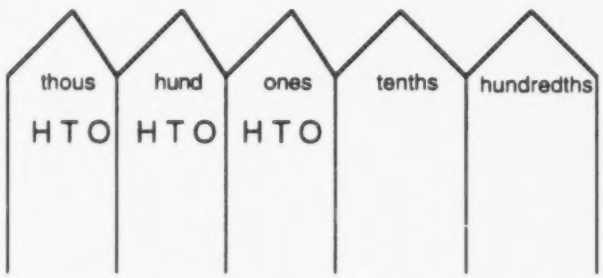
Another ordering appeared to treat decimal fractions that started with 0 as "below zero", as follows:

0.10 0.01 0.9 0.5 0 1 2 1.4 1.5 1.43

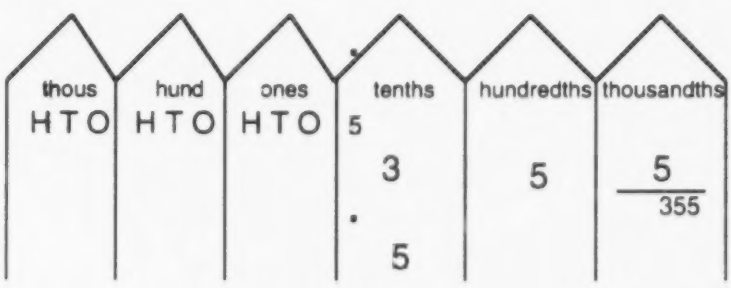
This student ordered numerals which started with 0 backwards ignoring the decimal point (with 0.01 and 0.10 probably being seen as the same). Numerals that started with 1 followed the standard forward sequence, again ignoring the decimal point. Note that all three of these examples have a system that pivots around 0 as equivalent to the decimal point rather than pivoting around 1.

For many students there was a tight bond between the numbers 0, 1 and 2. Years of counting had shown them that these numbers came next to each other. The students had had 3 or 4 years of instruction in decimals, and used them appropriately in many situations. However, on the pretest 63% of the students put 0, 1, and 2 next to each other at some point on the number line. On the two posttests, 58% and 50% respectively still put them next to each other. This indicates the strength of the whole number sequence and the difficulty of breaking into it in order to establish a meaning for decimal fractions.

The case in which students used "number tents", which a teacher had probably used to explain thousands and millions, is given below. They were asked to explain why 0.5 was larger than 0.355, despite the fact that 355 was larger than 5. First, they drew the following diagram:



One boy wrote 5 in the tenths column and 355 in the thousandths column (shown by the smaller numerals in the following diagram). The other boy then crossed out the 355, and wrote 355 spanning the tenths, hundredths and thousandths column (as shown by the larger numerals). It could be considered correct to put 355 in the thousandths column and that is how we say the fraction; it is only the need to fill every place in place value representation that lead to the other boy's placement of the digits.



These boys used a diagram intended for demonstrating the way in which large numbers are said and transferred this to explaining place value columns for decimal fractions without conflict.

This same pair became bogged down on whether or not there was a oneths column. It seemed necessary for symmetry around 0 or the decimal point, which came at the same place for them. The question leading to the discussion was whether a hundredth was written as 0.100 or 0.01. The former looked right, yet they had discussed 0.355 as 355 thousandths. If 0.100 was to be a hundredth, then there had to be a oneths column, because 100 had three places in it. Yet on the other hand, oneths didn't sound right. They eventually decided on the basis that 0.100 looked right, but their discomfort with this was exactly the unease that Piaget discusses as needing to be resolved to return to equilibrium. Adults would say that they reached the wrong equilibrium on that occasion. However their need to find such a justification suggested that they would fight a similar battle again. If the community around them gave them enough models and correction, they would learn that another solution is needed.

The students in this study were also actively working out how to operate with decimal fractions. Their debates were based on their knowledge of how counting numbers were added, subtracted, divided or multiplied as well as being based on a number of generalisations or rules that they had learned or discovered about operating with these numbers. The reason that students made more progress on contextual problems than on decontextualised problems was that their knowledge about quantities in the community provided a check on their answers. They wanted the answers to make sense when they were about money, amount of soft drink, or distance travelled. They also wanted to understand the answer that a calculator gave, in relation to their expectations. These chances to check the meaning of an answer were similar to the environmental checks that children have in learning language or counting.

A decontextualised problem that students checked against the calculator result was 0.9 + 0.1. Nearly all students predicted that this would give 0.10 because 9+1=10. The calculator then gave them 1.0 as an answer. The least sophisticated student ignored the decimal point when doing this problem, and was quite happy when the calculator read without that dot confirmed that 9+1=10. Other children were surprised and worked to see why the calculator said what it did. The best example came from a pair who had confidently predicted that 0.9 + 0.1 = 0.10. The pair were A and H; K refers to the researcher:

K:	Try it out [giving them calculator] [11 sec. pause - both surprised] What does it say?
H:	One point [pause]
K:	Why?
H:	Hmmm
K:	See if you can sort out between the two of you why it says that [6 sec. pause]
A:	I think that it's just making that as a zero
H:	Let's try it again clear it point 9 plus point 1 equals - point one, one point
	...
A:	That could be a zero, and um [16 sec. pause]

H: That's one point. That's one whole OH!! I know why now! Ten tenths is one whole! [pause]

A: Yeah!

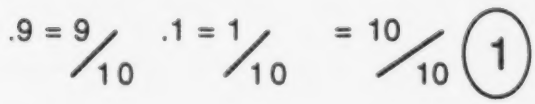
H: Ten tenths is one whole

A: Yeah

H: So obviously point nine and point one is ten tenths, that's one whole, where as, give me a piece of paper?

K: You can write on that

H: Point 9 equals nine tenths, and point 1 equals 1 tenth which equals to ten tenths, 1 whole:



Another problem in which there was conflict between expectations and calculator results was one in which students were asked to estimate how much soft drink would be left in a 1.5 litre bottle if you poured out a glass of 225 mL. Students could see the bottle and glass, and estimated that there would be more than a litre left. The problem was then what to put into the calculator. Many students wanted to subtract the smaller number from the larger number but even if they realised that 225 mL was smaller than 1.5 litres they had trouble knowing how to convey this to the calculator. L & V went through several attempts before they were satisfied with their answer on this problem. They, like others, preferred to work the problem out on paper to understand it. The workings of the calculator itself were opaque.

V: 225 takeaway 1.5

L: No.

V: Oh no!

L: 225 is lower than 1 point 5. 1 point 5 takeaway 225. One point 5 is 1 point 5 0, isn't it

L: [writes]

$$\begin{array}{r} 1.^{4}5^{1}0 \\ \underline{-2\ 2\ 5} \\ 1.\ 2\ \ 5 \end{array}$$

L: ... [pause] One point 2 5

K: Before you do it on the calculator, just what are you going to put in? What numbers.

L: [writes and says]

$$1.5 - 225 =$$

...

L: Equals - 2 thousand, two hundred and 3 point 5

V: (giggle)

L: Oh no, 2 hundred and

K: There's a minus sign in front of it. Do you know that? A negative [pause]

L: Oh [24 sec. pause] [adds to above so it reads]

$$1.5 - 225 = -223.5$$

(20 lines later)...

L: (very quietly) One litre is a thousand [22 sec. pause] [writes]

$$\begin{array}{r} 1\,{}^{4}5\,{}^{9}\!\!\not{0}\,0 \\ -\ 2\ 2\ 5 \\ \hline 1.\ 3\ 7\ 5 \end{array}$$

K: What are you doing there, L?

L: Taking 225 off 1 thousand 500

K: Why 1 thousand 500?

L: Cus one litre is one thousand, 1 point 5. I don't know if that's right.

Another case where it was hard to know where to put the decimal point came in a problem in which they were to add $4.95 and 90.9c as the cost of a meal at McDonald's plus the cost of the petrol to get there. As was often the case, the higher-ranked (M) went straight to addition while the lower-ranked peer (N) stayed close to the context and guessed more appropriately. Thus both partners had expertise to offer:

M: Ninety five dollars about, something cents... four cents.

N: Or five dollars.. cus its the whole thing.

M: Yeah I know, ninety five dollars and something cents Probably about ninety five dollars and four cents.

M: Where's the petrol?

M: That's how it costs all together isn't it?

N: Ninety five dollars?... for a McDonalds and petrol?

M: Yeah [laugh] oh

...

N: Five dollars, six dollars... six dollars point eight, no point, no Six dollars five.. uh six dollars five cents point nine

M: Six dollars five cents point nine?

N: Yeah

M: Point nine? [Pause] Six dollars and five cents point nine

N: Yeah.

M: That means six point five point nine

N: Oh man nah...

... [when about to use the calculator]

M: I have a weird feeling about this.

K: What's your weird feeling M?

M: I think it's going to come out as ninety five dollars and something

...

N: Five [writes]

$$4.95 + 90.9 = 95.85$$

...

N: 4 point ninety five plus ninety nine cents.
M: Four point ninety five plus.
N: Ninety nine cents.
N: Ninety nine cents. Ooh [N puts in 495+99=]
M: How did we get?
N: Um I didn't put the point

These students changed 90.9 c to 99 c because they knew how to operate with that. One of them repeatedly talked of rounding, but they did not round to 91c. Still, their answer gave them something which seemed reasonable, whereas $6.05.9 did not seem reasonable. In both of these last examples students solved the problem by getting rid of the decimal point, but only after they had struggled with its meaning.

Another rule that some students had learned was that when you multiply by 10 you add a zero. One problem for which they used this rule was either the contextual 10 x 1.5989 (changing NZ dollars to Samoan tala), or the decontextualised $1\frac{1}{2} \times 10$. A frequent answer to the first problem was $1.59890 and to the second 1.50. Other answers were $10.5989 and 10.5 accompanied by explanations that 1 times 10 was 10 and then you added on the other bit. When students did either of these problems on the calculator, a conflict situation again arose. Here are B and G:

G: 1 point 5 9 8 9 times 10
B: ...89 times 10 [both continue to say numbers]. Look how it all goes zeros across the line and you go point in the zeros.
G: Yeah point zero ...9 8 9 5 point 1:

$$\begin{array}{r} 1.5989 \\ \times \quad 10 \\ \hline 0.000 \\ 1.59890 \end{array}$$

B: Ok. Now "answer" is 1 point 5 9 8 9 0 That's our prediction
G: It's exactly the same.
B: Eh? What?
G: It's exactly the same.
B: Yeah I know. No, cus you added a zero actually. [pause]
G: Huh, Added a zero.
...
K: What does that mean, in terms of tala? Tala is really just the same word as dollar
G: A dollar 59? Woah!
B: Nah [10 sec. pause]
...

K: Sometimes if you just use your common sense, and don't think about the numbers, you are better off. About how much?

G: Shall we try - [pause] 12 dollars? [pause]

...

G: Yeah 10, 15 dollars, 13 to 15.

...

B: [writes]
1. 1.59890
2. $11 to 15

...[They get a calculator answer of 15.989]

G: Yeah, see the zero DOES go here. You multiply by 10 and the zero goes on, and then (pause) the 5 goes there, should be a 0 on here, Should be a zero here, a zero there because you are multiplying by 10, so you'd get a different answer now.

This pair does not appear to fully understand why the answer is 15.989. But at least they have accepted their original answer, involving addition of a 0, was wrong.

Another interesting example was provided as A and R tried to work out $1\frac{1}{2} \times 10$:

A: Fifteen. What do you predict?

R: Ten point 30 [writes]

A	R
15	10.30

R: Half is thirty, and what's 10 times one? Equals 10

K: Can you two agree?

A: Well, just think about it in your head, don't worry about that, but one, one times ten equals 10 and half of one, which is point five, two halves equals one, so it's five, so 10 and 5 is 15. [3 sec pause]

R: Ok

...

K: Ok. But there's a difference, ... between 10.30 and 15?

R: Oh I did, I put 10 times 1 equals ten and a half, that's 30.

K: uh huh

R: I was thinking whole numbers

A: You must be thinking about time

R: Yeah

A: Cus half of 60 minutes makes a whole hour in time so half of that it's 30, but in money and real life it's 50

R: Oh yah

Summary

None of the errors made by these students were random. They appeared to be thoughtful attempts to solve the problems they had been given. Students checked their estimates and their results against other knowledge and were uncomfortable with answers that did not make sense. Whenever there was a context, they used it – they weren't going to pay $95 for a trip to McDonalds! They were interested in a calculator's answer but needed to make sense of it, and usually used written calculations for this purpose.

Some of the problems they were given may seem unusual. This was intentional. When students deal with dollars and cents, always given to two decimal places, they can use whole number procedures and be correct. To understand what the decimal point means they needed the challenge of money to one, three or four decimal places. In these dialogues they were able to go beyond the usual, highly decontextualised classroom discourse. They engaged in constructing meaningful mathematics rather than have this meaning be what Cobb, Yackel & Wood (1993) describe as an accidental by-product of instruction.

The dialogues also suggest the myriad of ways in which students can get things wrong. For example, the diagrams which teachers use make sense to adults because they fit an adult's image of the number system. If a student is still constructing an image of this system, they are not necessarily helpful.

Decimal fractions are difficult precisely because students come to them with a rich collection of ideas about numbers and how they work. They then need to reconstruct these ideas to include decimal fractions. Our challenge as educators is to give them enough chances to check which of their ideas work in this new part of the number system, and which ideas require adaptation.

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Chapter 14

Mental Computation and Number Sense of Western Australian Students

Alistair McIntosh

Introduction

Over the past 20 years the opinion has been increasingly expressed (McIntosh, 1979; Plunkett, 1980; Reys, 1984; Trafton,1986) that written computation plays too large a part in the mathematics curriculum of the primary school, while mental computation is comparatively neglected. Specifically the emphasis on the teaching of written computation is criticised because it is little used by adults (Wandt and Brown, 1957), is mistrusted and shunned by children (Carraher, Carraher and Schliemann, 1987), gives false ideas about mathematical activity (Hope, 1987) and is non-constructivist (Kamii, 1989); the neglect of mental computation is equally inappropriate since it is used for most calculations by adults (Maier, 1980), is the simplest way of doing many calculations (Hope, 1986) and is an excellent way of learning how numbers work (Ewbank, 1977).

In a series of studies conducted in Brazil, Nunes, Schliemann and Carraher (1993) reveal clearly the dichotomy between the largely self-taught oral computation strategies used by children and adults at work and the rule-bound written methods taught and used often by the same subjects in school. They show that, compared with written school procedures, the oral methods were better understood, were more reliable, achieved more accurate results, gave results which, when wrong, were closer to the correct answer, and remained closer to the meaning of the original problem. Thus it was rare with the oral methods, and common with the

written methods, for the subjects to be unsure which operation was needed to solve the problem and for the answers to be quite inappropriate for the question posed.

However, the teaching of formal written algorithms is still seen as the central core of the primary school mathematics program by most primary school teachers, is expected by parents, and is taken for granted as a major task of the primary school by society at large. A shift in emphasis from written to mental and oral computation is necessary if more appropriate and effective methods of calculation are to be used. Although some research has been conducted examining the use of children's computational strategies, more research appears to be a necessary and responsible forerunner to any major reforms in teaching and curricula.

Some work has been done on the mental computation strategies of primary aged children, but often by psychologists more interested in using it as a tool for examining the role of memory than as the result of any curricular interest, and almost all research has been confined to mental strategies for the basic facts. Very little exists in the literature regarding the development of number sense of young children to students in the middle to upper primary grades. Some Australian studies (Wright, and Cooper et al., this volume) are focusing attention on counting and computational strategy development but more extensive research needs to examine the role of mental computation as part of developing number sense generally.

Number sense

It is more difficult to define number sense than to recognise its presence or absence in particular incidents. In a list of behaviours which "demonstrate some presence of number sense" Sowder (1993) includes the ability to compose and decompose numbers, to move flexibly between different representations, to recognise when one representation is more useful than another, to recognise the relative magnitude of numbers, to deal with the absolute magnitude of numbers, to use benchmarks, to link numeration, operation and relation symbols in meaningful ways, to understand the effect of operations on numbers, to perform mental operations through "invented" strategies, to use numbers flexibly to estimate and to recognise when an estimate is appropriate, and, finally, a disposition toward making sense of numbers.

Clearly these abilities overlap and some are much broader than others. McIntosh, Reys and Reys (1992) have attempted to analyse the components of number sense and to arrange them in a hierarchical framework. It is generally agreed that the ability to compute mentally in flexible ways is both a component and an indicator of number sense. It is tempting to speculate that the two may correlate closely with each other.

In 1989 Ellita de Nardi, Paul Swan and I started investigating the mental computation strategies of students in grades 2 to 7, in order to gain greater awareness of the range of these strategies. The purpose of these inquiries was to underpin subsequent curriculum development aimed at providing more relevant and efficient mental computation activities for primary teachers.

In 1991 I was invited by Professors Robert and Barbara Reys of the University of Columbia-Missouri to include Western Australia in a comparative study of the mental computation abilities of Japanese and American students in grades 2 to 8. The Western Australian part of this study was conducted in cooperation with Jack Bana and Brian Farrell of Edith Cowan University.

Since 1993 the study has enlarged its scope to consider the number sense of the same cohorts of students in Australia and the United States as were included in the study of mental computation. This study has led to international cooperation between Edith Cowan University and the University of Missouri-Columbia in the development of an analysis of the components of number sense, the development of a classified item bank of number sense test items, and comparative data on many mental computation and number sense items from the United States, Australia, Sweden and Taiwan.

The following pages give a brief account of three research projects conducted by the author with a variety of colleagues over the past six years in the fields of [mental computation and] number sense of students in grades 2 to 9, together with some consideration of the implication for teaching.

Study 1 - *Mental Arithmetic Project* (MAP)

Description of the study

The Mental Arithmetic Project had two major goals: first, to discover, describe and analyse the mental arithmetic strategies of primary age students, second, to devise and trial classroom practices which might lead to more awareness of these strategies among teachers, and more proficiency and facility in flexible mental computation by students. Only that part of the project directed toward the first goal is described here. Material designed to achieve the second goal is described in McIntosh, De Nardi and Swan (1994).

Three primary schools were selected to provide a variety of situations in the metropolitan area; they consisted of a large state primary school, a Catholic primary school and the junior department of a boy's private school. In each school, one class at each grade level from 2 to 7 was randomly selected (except where only one class at each age existed). The class teacher of each class was asked to select four children for the study: two of "average" mathematical ability, one of "above average but not exceptional" ability and one of "somewhat below average" ability.

Clinical interviews were conducted with each of the 72 students. Each student was interviewed three times over two months, during which a total of 45 mental computation

calculations involving whole numbers was given orally. The same 45 items were used for all children in all grade levels. After each calculation the student was asked to describe how the calculation had been performed. Notes were taken during the interviews of visual cues such as the use of fingers and eye or head movements. All interviews were audiotaped, transcribed and analysed using a classification system devised and refined by the researchers during the study.

The 45 items were chosen to reflect the range of calculations considered by the researchers to be within the scope of at least some of the students in the survey. Several items were also included which the researchers considered unlikely to have been given as mental calculations to students of this age but which were considered nevertheless to be possible for some students. In the event this assumption was justified. With three exceptions (items 22, 43 and 44) whole number items only were selected in order not to automatically exclude younger students for whom the study of fractions or decimals had not yet begun.

Four items (Items 9, 14, 37, 38) were included as an indicator of the students' short term or working memory (STM). A 3-, 4-, 5-, and 6-digit number was presented orally to the student, whose task was to repeat the numeral backwards. The longest string correctly answered was taken as a measure of the student's STM.

Results and discussion

Overall results

Table 14.1 shows the number of children (maximum 12) at each grade level correctly computing each item mentally. The items are grouped into clusters of linked items. The numbering of the items corresponds to the order in which the questions were asked, in three groups of 15 questions on three occasions. Some items are repeated because the same question was asked on two occasions for the purpose of comparison.

Strategies Used

Analysis of the over 3000 responses (72 children x 45 items) revealed the following strategy clusters. The full list of strategies with examples of children's responses using these strategies is contained in McIntosh, De Nardi & Swan (1994).

I *Initial strategy:* Used commutativity, or turned subtraction into addition, division into multiplication.

C1 *Elementary Counting:* Counted forward or backward in ones

C2 *Counting in Larger Unit:* Ranging from counting on or back in twos to reciting tables.

P1 *Used Place Value Instrumentally:* Using Skemp's terminology (Skemp 1976), used place value mechanically: for example removing zeros without knowing why it worked, and calculating mentally by recreating the standard written algorithm.

Table 14.1
Results by grade and item cluster.

No.	Problem	Gr 2	Gr 3	Gr 4	Gr 5	Gr 6	Gr 7	Total
1	2 + 7	9	11	11	12	12	12	67
10	9 - 7	9	11	11	12	12	12	67
6	20+70	8	10	12	12	12	12	66
4	90 - 70	6	11	12	12	11	12	64
2	8 + 6	9	9	12	10	12	12	64
8	14 - 6	8	10	11	12	12	11	64
31	14 - 6	7	10	11	12	12	11	63
12	80 + 60	3	9	11	11	11	11	56
15	140 - 60	2	8	9	11	12	12	54
23	2 x 7	7	12	11	12	12	12	66
17	14 ÷ 7	9	8	10	12	12	12	63
35	14 ÷ 7	8	9	10	12	12	12	63
16	5 x 3	6	10	12	12	12	12	64
21	15 ÷ 3	5	6	11	12	12	12	58
36	150 ÷ 30	4	4	8	10	12	12	50
30	150 ÷ 30 story	4	3	7	10	12	11	47
26	150 ÷ 30	3	1	9	7	10	12	42
39	25 + 8	7	12	10	11	11	11	62
40	25 + 13	5	11	9	12	11	12	60
41	25 + 27	4	8	9	11	10	12	54
42	25 + 98	3	5	7	11	8	11	45
25	4 x 9	4	4	11	12	12	12	55
19	36 ÷ 9	4	5	7	10	12	12	50
24	360 ÷ 9	0	1	4	7	11	12	35
34	8 x 6	2	4	8	12	9	12	47
27	48 ÷ 6	0	3	7	12	11	12	45
18	8 x 6	2	4	6	10	10	12	44
29	8 x 60	0	2	3	10	11	11	37
36	150 ÷ 30	4	4	8	10	12	12	50
30	150÷30 story	4	3	7	10	12	11	47
26	150 ÷ 30	3	1	9	7	10	12	42
4	90 - 70	6	11	12	12	11	12	64
15	140 - 60	2	8	9	11	12	12	54
32	74 - 30	4	5	8	9	6	10	42
7	80 - 24	1	3	6	8	7	7	32
11	74 - 30	1	1	6	9	9	5	31
13	114 - 30	1	2	2	5	7	5	22
23	2 x 7	7	12	11	12	12	12	66
45	double 39	3	6	10	12	11	11	53
22	half of 30	2	5	8	11	11	11	48
43	a quarter of 120	0	3	7	9	5	10	34
44	half of 132	0	2	4	5	7	11	29
20	$3 + 25c story	3	4	7	11	10	10	45
28	$3 + 35c story	0	1	2	4	3	7	17
9	reverse 283	9	7	10	11	9	11	57
37	reverse 5387	3	5	8	6	10	8	40
14	reverse 29528	0	0	1	4	3	7	15
38	reverse 748592	0	0	0	0	3	2	5

P2 *Used Place Value Relationally:* Any use of place value and notation which showed understanding. Includes bridging tens or hundreds or otherwise using tens and hundreds, working from the left or right and adding or subtracting parts of one number.

DH *Used Doubling or Halving*

F *Used Fingers*

K *Known Fact:* Gave answer instantaneously or otherwise demonstrated prior knowledge of the answer

RK *Related to a Known Fact:* Related calculation to another calculation whose answer was known.

Some earlier studies had tended to describe a solution strategy by a single label. It was our experience that even with simple basic fact solution strategies, students often used a variety of strategies sequentially in the one solution. We decided therefore to use a flexible multiple coding system. As a result solutions were coded as employing between one and four strategies.

As an example, here is Andrew calculating 36 ÷ 9:

I went 9, 18, 27, 36 and counted up.

The coding for this solutions was I, C2, F, since Andrew changed the division into a multiplication (I), solved the multiplication by counting up in nines (C2), and was observed keeping a record of his skip-counting on his fingers (F).

Findings related to age

Table 14.2 shows the mean percentage score at each age level on the 41 items excluding the STM items.

Table 14.2
Mean Percentage Score on MCT by Grade Level

Grade	2	3	4	5	6	7
Mean Score (%)	34.6	52.0	74.0	89.8	90.0	96.1

Not surprisingly, the average number of items answered correctly increased with age. However the variations between children within the same age range (though they had been selected to exclude the extremes of ability) was noticeable.

While there were only 6 questions to which none of the youngest age group responded correctly, there were 16 items which one or more of the youngest age group calculated correctly whilst one or more of the oldest age group (five years older) did not.

Findings Related to Ability

From the 12 children in each grade group, the three least competent and the three most competent children were chosen for comparison. The children were selected on the basis of their answers to the MCT. Comparisons were then sought between the least and most competent mental calculators.

The average scores of the two groups of students on the short term memory items are shown in Table 14.3. If the child failed to reverse the shortest (three-digit) string correctly a score of two was assigned. Thus the score could range from 2 to 6.

Table 14.3
Average STM Score (most/least competent)

Grade	2	3	4	5	6	7	All Grades
Most Competent	2.3	3.0	3.7	3.7	5.0	5.0	3.8
Least Competent	3.0	3.7	3.3	3.3	4.0	4.0	3.6

There is no indication here that short term memory is a decisive factor in the superiority of more competent mental calculators.

Table 14.4 shows the overall number of types of strategies used by the two groups (L = least competent, M = Most competent).

Table 14.4
Strategies used by grade and category(least/most competent)

	Gr2		Gr 3		Gr 4		Gr 5		Gr 6		Gr 7		Total	
	L	M	L	M	L	M	L	M	L	M	L	M	L	M
I	5	19	8	21	15	35	28	27	37	22	36	23	129	147
C1	8	9	18	15	8	5	12	5	10	3	6	1	62	38
C2	1	26	10	19	9	21	14	3	13	8	13	3	60	80
P1	2	2	2	6	7	16	22	22	16	21	16	26	65	93
P2	1	25	4	17	19	34	18	44	28	35	34	27	104	182
DH	1	4	1	3	5	5	7	4	2	3	9	3	25	22
F	12	15	11	9	8	1	12	1	4	2	5	2	52	30
K	2	3	1	7	3	15	12	25	14	29	15	40	47	119
RK	0	7	2	15	8	26	19	27	23	25	31	17	83	117

Care must be taken in interpreting the data in this table. It might appear that, in the older grades, the less competent calculators used more strategies than the more competent. However this is largely because the more competent knew or could infer more of the results (K, RK) and therefore did not need to have recourse to other strategies. Nevertheless, some observations may be made:

1. Throughout Grades 2 to 7, the less competent make more use of C1 strategies; that is, they rely more heavily throughout on counting on or back in ones. In contrast, the more competent make much more and much earlier use of C2 strategies.

2. More competent children use particular strategies much earlier than less competent; for example, the use of P2 strategies by the competent in grades 2 - 4 (76 occurrences) is as great as the use of these strategies by the less skilled in grades 5 - 7 (79 occurrences).

3. The skilled make rather more use of P1 strategies. Analysis shows that almost all the uses noted here refer to removing the zero (only 29 instances of the use of the written algorithm were recorded, all of them in grades 5 to 7). It may be that in some cases they were using the strategy with understanding; however the protocols show that in most cases they were operating with the mechanical rule.

4. Except in Grade 7, by which time they gave an immediate correct response to most of the items, the more competent children made much greater use of P2 strategies, that is, they were able to manipulate numbers and exploit the place value aspects of them in a dynamic way.

5. There is not a consistent difference between the two groups in their use of doubling and halving; this strategy appears to be used by both groups and its use is relatively stable throughout grades 2 to 7.

6. Except in Grade 1 the less competent make more use of their fingers as a counting aid.

In general then the more competent start using particular strategies earlier, make more active use of place value strategies, and know more facts. The less competent gradually make use of more strategies but rely more on elementary counting and fingers than do the more competent.

Study 2 - *Mental Computation Test/Western Australia* (MCT/WA) *Study*

Description of the study

This study was designed to provide three different perspectives on the mental computation of Western Australian students in grades 3, 5, 7 and 9: first, a survey of the kinds of computations which they prefer to do mentally; second, a measure of attitude towards mental and written computation; and third, an assessment of their performance on mental computation items.

Three different survey instruments were developed for the study: a Preference Survey (PS), an Attitude Survey (AS), and a Mental Computation Test (MCT). All the instruments were administered (in that order) during one 50 minute class period for all grade levels 3, 5, 7 and 9, except that the Attitude Survey (AS) was not used in grade 3.

A typical metropolitan secondary school participated in the study together with three of its major "feeder" primary schools (grades K-7). Within each primary school, two classes were randomly selected at each of the grade levels 3, 5 and 7. In the secondary school, where students were streamed on ability, stratified random sampling was used to select three pairs of classes, with each pair at a different level of ability.

The Preference Survey (PS) consisted of a series of numerical computations. Subjects were not asked to carry out the computations but only to decide if they would do each computation mentally if allowed a choice between calculating mentally, with pencil and paper, or with a calculator. Students indicated their answer by marking "yes" or "no". Some items in the PS were also included in the MCT so that data reporting preferences could be compared with actual performance on the same item.

The Attitude Survey included 28 statements clustered by five dimensions: Interest and Enjoyment, Perception of Competence, Perception of Value, Perception of Use, and Perception of Source of Instruction. Typical examples to which the subjects were asked to respond were:

I think I will do written computation more than mental computation as an adult
(Perception of Use)
I learned to do mental computation at school
(Perception of Source of Instruction).

The Mental Computation Test was designed by the researchers for group administration. The grade 3 and 5 MCT versions contained 30 items, 15 administered orally and 15 administered visually. The Grade 7 and 9 versions contained 40 items, 20 administered

orally, and 20 administered visually. Table 14.5 shows the distribution of items by operation and number type for each grade.

Table 14.5
Mental Computation Test Item Distribution

Number Type	Operation	Grade 3	Grade 5	Grade 7	Grade 9
Whole Numbers	Addition	12	6	4	-
	Subtraction	12	6	4	-
	Multiplication	4	6	6	-
	Division	2	6	6	-
Fractions	Addition	-	2	4	4
	Subtraction	-	2	4	4
	Multiplication	-	-	2	4
	Division	-	-	-	2
Decimals	Addition	-	2	2	1
	Subtraction	-	-	2	2
	Multiplication	-	-	2	2
	Division	-	-	-	2
Percentages	Multiplication	-	-	4	5

Results and discussion

The Preference Survey revealed some interesting clues to students' conceptual understanding or lack of it. For example, more than 40 per cent of grade 5 students would not do the item 100 x 35 mentally and between one third and one quarter of grade 7 and 9 students would not calculate mentally 945 x 1000. One third of grade 7 students would prefer not to calculate 1 - 1/3 mentally, and over half would not mentally calculate 0.1 x 45.
The Attitude Survey revealed that, in general, students felt they did more written computation than mental computation at school, but that the reverse would be true out of school; these views became somewhat more pronounced for the older students.

In the Mental Computation Test, only in Grade 3 did a *t*-test show a significant difference ($p < 0.5$) between the mean scores on items presented visually and orally (5.7 visual and 6.75 oral). However parallel research with Japanese students shows that the mode of presentation can significantly affect performance (McIntosh, Reys and Reys, 1995). In addition it is suggested that the mode of presentation does affect the range of strategies used by students. When students can see the calculation they are more liable to use mental forms of written computation, whereas only hearing the calculation appears to encourage more flexible solution strategies.

In the Preference Survey students were presented with a number of computations and asked whether or not they would prefer to do them mentally. At each grade level, a sub-section of these items was also included in the MCT. If we assume that those students who stated that they preferred to do the calculation mentally thought they would calculate the answer correctly, then for each of these items (nine items for each of grades 3 and 9 and ten for each of grades 5 and 7) it is possible to classify all students into four categories:

1. Those who thought they could do the calculation, and could.
2. Those who thought they could do the calculation, but couldn't.
3. Those who thought they couldn't do the calculation, but could
4. Those who thought they couldn't do the calculation, and couldn't.

Analysis of the results suggests that boys are somewhat more likely than girls to be justifiably sure of their ability to calculate mentally (category 1), while girls are somewhat more likely to be wrongly diffident of their ability (category 3). Moreover, while category 1 students were naturally more likely to be in the upper quintiles of ability, and category 2 and 4 students more likely to be in the lower quintiles, students in category 3 were spread much more evenly across the ability range. This appears an interesting area which merits further investigation.

Study 3 – *Number Sense Test/ Western Australia* (NST/WA) *Project*

Description of the study

This project had two main aims: first, to investigate the possibility of assessing number sense by group tests; and second to gain some information regarding the number sense of students at grades 3, 5, 7, 9.

Four group tests of number sense were compiled, one for each of the four age levels 3, 5, 7 and 9. Some questions were common to two or more age levels. The majority of the questions used a multiple choice format, but others asked for open answers, for written explanations or for marks to be placed on number lines or other displays. All items were scored either "correct" or "incorrect".

The tests were administered to the same classes of students as were involved in Study 2 described above; the overlap of students in the two studies is shown in Table 14.6:

Table 14.6
Table showing the overlap of student participation in the MCT/W and the NST/WA studies

	Participated in MCT/WA only	Participated in NST/WA only	Participated in both studies
Grade 3	3	20	160
Grade 5	8	12	155
Grade 7	10	15	153
Grade 9	35	7	117

The group tests of number sense were compiled by the researchers and refined after trialing. The final tests administered in the study contained 30, 35, 45 and 45 items for grades 3, 5, 7 and 9 respectively. Items were selected from those described in the literature or devised by the researchers. Items were selected to match conceptual knowledge appropriate to the grade levels as described in the State mathematics syllabus and to accord in general terms with the number sense framework described in McIntosh, Reys and Reys (1995). The tests were presented in booklet form, with items on the left hand side of the page and spaces for answers on the right hand side. No time limit was placed on the test.

Results and discussion

Analysis of the test results is still continuing. The following are some preliminary results and comments.

It appears that the tests were appropriately matched to the age levels. Only one student (in grade 9) achieved the maximum score, while all students were successful on at least three items, with a mean success rate of over 50% at each age level.

Using the framework for analysing number sense proposed in McIntosh, Reys and Reys 1992, we can consider the results under three headings: number, operations and applications. This summary will be restricted to comment on the first two categories.

Whole numbers

In general all cohorts showed a reasonable sense of the base ten number system applied to whole numbers. For example 51% of Grade 5 students and 70% of Grade 7 students gave the correct response, 37, to the following question.

If I have $378 in my savings account and withdraw all my money, how many 10-dollar notes would the bank be willing to give me?

The extent to which wrong answers were due to imperfect understanding of our number system as opposed to misinterpretation of the question would need to be ascertained by error analysis and clinical interviews.

Decimals

Only 16% of Grade 7 students, and 62% of Grade 9 students, said that there were "lots" of decimals between 1.52 and 1.53 and were also able to name two; 42% and 23% of the cohorts respectively thought that there were "none". More seriously misconceptions were revealed by the following item:

Circle the decimal which best represents the amount of the box shaded.	A 0.018 B 0.15 C 0.4 D 0.801 E 0.52

Only 51% of Grade 7 students and 74% of Grade 9 students gave the correct response, B.

Fractions

The tests showed that students' conceptual grasp of fractions was much weaker at all levels than their understanding of the decimal system. For example, 57% of Grade 5 students thought that 5/9 represented a larger fraction than 5/6, and 57% of Grade 7 students and 29% of Grade 9 students thought no fractions existed between 2/5 and 3/5. Only 35% of Grade 9 students could name a fraction between 0 and 1/10 whose numerator was not 1. Asked to circle fractions which were greater than 3/4 but less than 1, 42% circled 2/3, 23% circled 7/10, and 7% circled 4/3. Only 47% of grade 5 students and 56% of Grade 7 students could name a fraction between 1/2 and 1 other than 3/4. Only 34% of Grade 5 students and 56% of Grade 7 students could place 1/10 and 4/5 on a number line marked from 0 to 1, within acceptable limits. Many marked 1/10 to the right of 4/5. Asked to circle statements that are true about the number 2/5, 25% of Grade 7 students and 17% of Grade 9 students circled "It is the same as 2.5".

Sense of operations

Items concerning the effect of multiplying and dividing by fractions and decimals revealed widespread misunderstandings. Only 53% of Grade 7 students thought that 29 x 0.98 was less than 29, while 30% of Grade 7 students thought that 0.5 x 840 was the same as 0.50 x 84. Also 18% of them thought that 292.5 was a "reasonable" answer for 4.5 x 6.5. 30% of Grade 9 students thought that the best estimate for 54 + 0.09 was "a lot less than 54". 73% of Grade 7 students and 49% of Grade 9 students thought that 29 + 0.8 was less than 29.

Only 48% of Grade 9 students were aware that 456 ÷ 8 is the same as (rather than "greater than" or "less than") 456 x 1/8.

Table 14.7 reveals some misunderstandings regarding equivalent fractions.

Table 14.7
Distribution of responses to NST test item by grade 7 and 9 students

Circle the number you can put in the box to make this sentence true:			Responses (%)	
			Grade 7	Grade 9
$\frac{1}{2} \times \square = \frac{3}{6}$	A	$\frac{2}{4}$	8	10
	B	$\frac{2}{3}$	28	13
	C	1	28	46
	D	3	34	31

One can imagine how easily the instruction "multiply top and bottom by 3" can be equated with "multiply the fraction by 3".

How many days have you lived?

One question provided such surprising data that the results are given in full. Table 8 gives the item as presented together with the percentage at each of the four age levels giving each of the four responses.

Table 14.8
Distribution of responses to NST test item.

Item	Possible responses		Responses (%)			
			Gr 3	Gr 5	Gr 7	Gr 9
About how many days have you lived? (Circle the nearest answer.)	A	300	31	7	1	1
	B	3000	34	35	38	32
	C	30 000	24	37	48	54
	D	300 000	11	21	13	13

Many of the 31% of grade 3 students who responded as if they were less than one year old may not have known the number of days in a year. That about half of the Grade 7 and Grade 9 students chose the equivalent of over 80 years, or that over 10% at each age level suggested they had been born in the 12th century A.D., are decisions less easy to explain.

Implications for teaching and research

In this chapter it has not been possible to give more than a glimpse of some of the aspects of this research program. In particular the rich evidence of children's mental computation strategies revealed by the interviews is not represented here, nor the very heartening evidence from the comparative study of mental computation which showed that the Australian sample, while far behind the performance of the Japanese sample at grade 3, did at the grade 9 level overtake them. Moreover we have only hinted at the wealth of evidence of misunderstandings, particularly of fractions and decimals even at the grade 9 level, revealed in the number sense tests.

The conclusions given below are drawn from complete data rather than from the data given in the current chapter.

It is clear that at present very few children acquire the range of mental computation strategies they possess as the result of deliberate classroom interventions or practices. And yet, sooner or later, most children acquire many or even most of them. For the vast majority of primary school children the full range of strategies is potentially available: Not one single strategy was the province of the more competent only.

Much exploratory work is needed however in deciding how best to spread these strategies around. At present insufficient work is directed toward this end. A great deal more support has been given recently, partly as a result of the *National Statement*, the *Profiles*, and a number of State and local initiatives, to the teaching of mental computation and in particular to class discussion of the strategies used by children. However it is probably true to say that the majority of primary teachers are not aware of the range of efficient mental strategies which children can acquire, let alone how best to help children acquire them. We need a great deal of action research in this area, carefully planned and documented.

One thing does stand out from the interviews with children with regard to the teaching of mental strategies. Teaching rules (for example "removing zeros") is as dangerous and self-defeating for mental computation as it is for written computation. This was the only strategy which children stated (and they stated it frequently) had been taught them by parents and/or teachers; and it was the only strategy which they misused through lack of understanding. There is a powerful lesson to be learned here. Helping children to acquire ownership of efficient strategies is not likely to be achieved by "teaching" these strategies at the expense of

understanding. We must not make the same mistakes over the teaching of mental computation as we have made with regard to the teaching of formal written algorithms.

Even when children devise their own strategies we must expect times when these strategies are only partly formed: Children need to feel free to experiment and to verbalise their strategies. Sometimes the act of verbalising illuminates an error for the child. At other times it provides us with an insight into the process so that we can intervene effectively.

For most mental calculations there is not "one" strategy, nor is there a "best" strategy. It would be difficult to find a calculation for which all children who produce the correct answer use the same strategy. Moreover to suggest that for this or that calculation there is a "most efficient" way seems to us, even if it may be true in some isolated examples (for example 99 + 28), to be giving exactly the wrong message about how to become proficient at general mental calculation. The efficient mental calculators we interviewed appeared to have a range of strategies which they used and adapted flexibly. They did not necessarily perform the same calculation the second time around in the same way as on the first occasion; but they did show a command of relationships between numbers which they orchestrated efficiently in order to perform a calculation with the least mental strain. Thus they did not need a "better" short term memory; they simply put fewer demands on it.

A crucial feature in opening up simple and more efficient mental strategies is an operational understanding of place value and ownership of its essential features. As an obvious example, children who add ten by counting on ten ones have set themselves a more difficult task and are therefore more prone to error than children who are aware that adding one to the tens digit achieves the same end.

It would appear that competent mental calculators become so in spite of what happens in the classroom; or, perhaps more likely, they possess or acquire an early affinity with numbers which allows them to "play" with them and they abstract from classroom practices of any kind skills and understandings which they adapt and use in mental computation and any other mathematical situations. The children who should concern us most are the less competent and less confident ones who do not receive any deliberate help from current classroom practices and do not abstract these understandings and skills for themselves. As a result they are left stranded with primitive and less efficient strategies. It is important to expose them to more sophisticated strategies partly because these are often, in fact, simpler. How best to expose them to these strategies is not yet clear.

The present number sense study was primarily concerned with testing the usefulness of group paper-and-pencil tests in providing helpful information about the number sense of students. While we are optimistic that they can provide valuable data, there is little doubt that clinical interviews with individual students are essential to provide richer detail about the thinking of students.

The current number sense data make it quite clear that, while the Australian sample has a reasonable understanding of whole numbers (at least up to 1000), its understandings of fractions, decimals and percentage and operations involving them are often totally inadequate for their future needs. If at the end of the primary school stage you see no difference between 2.5 and 2/5, would not care to calculate 1 - 1/3 mentally and are hazy about the relative size of decimals, you are well on your way to being an innumerate adult.

We must learn, in the face of these facts, not to blame the students, but to query our teaching goals and our teaching methods. We must start to take on board the accumulating evidence of research as to what our students can and cannot do as a result of present practices and to plan our syllabus documents, our classroom practices and our assessment systems accordingly. Certainly force-feeding students undigested facts and skills is not an efficient or appropriate way to prepare our students for the mathematical demands of present day society.

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